

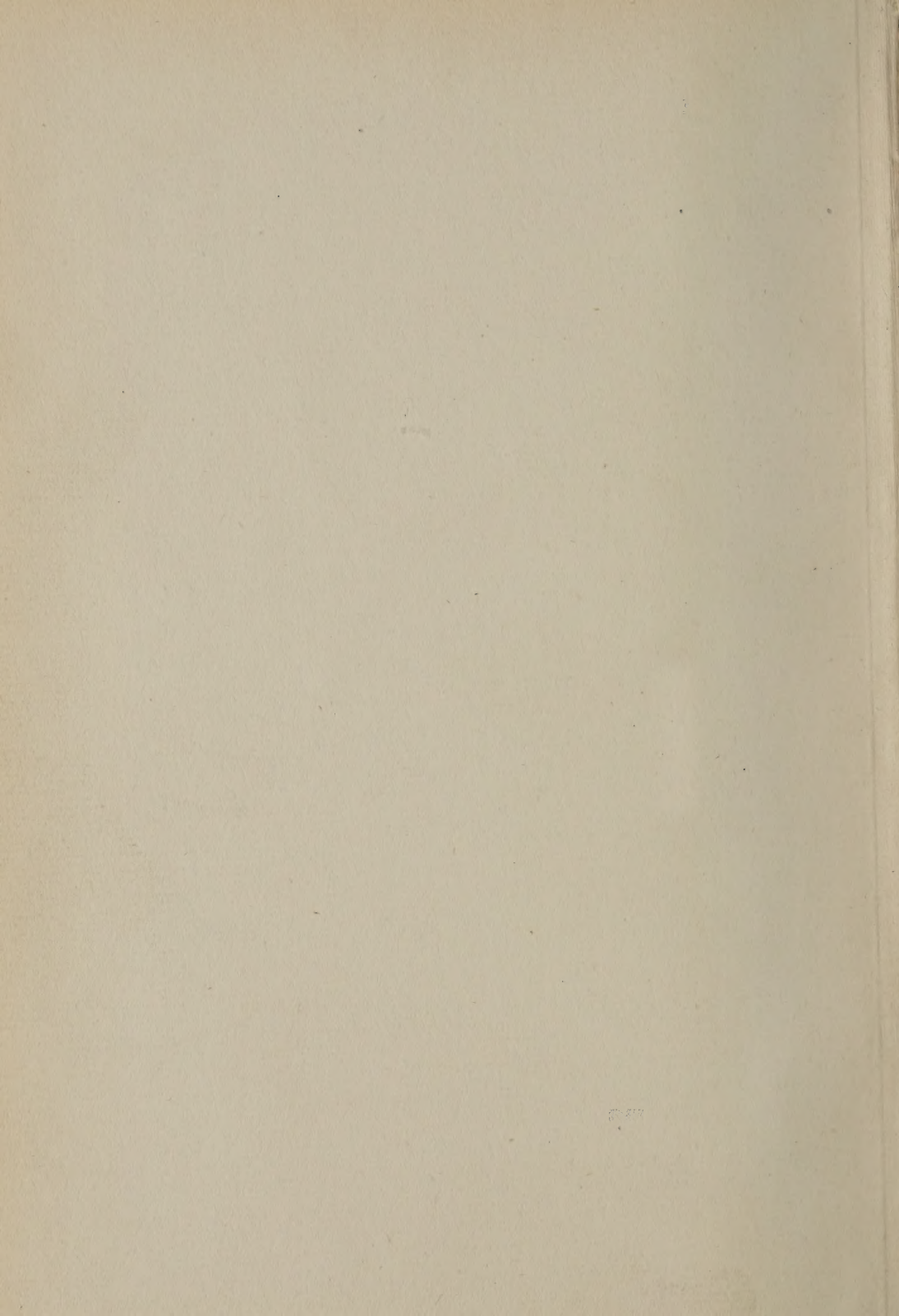


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Some Circular Curves Generated by Pencils of Stelloids and Their Polars,

by

CLARENCE MARK HEBBERT, Champaign, Ill., U.S.A.

I. Introduction.

It is the purpose of this paper to consider the transformation $z' = \frac{1}{z^2}$, which has the three cube roots of unity for double points and with which is connected the pencil of stelloids (cubics)⁽¹⁾ through the three cube roots of unity and their associates. Some properties of the quintic generated by the pencil of cubics and the first polar pencil (equilateral hyperbolas) will be derived.

The more general transformation $z' = \frac{1}{z^n}$ will also be studied and the general form of the product of the pencil of stelloids through the $n+1^{\text{th}}$ roots of unity and their associates, and the first and second polar pencils of any point (x', y') will be determined. Some properties of the asymptotes and foci of these curves will be derived. This transformation is simply the contracted form of the general transformation $z' = z - \frac{(n+1)f(z)}{f'(z)}$, for $f(z) = z^{n+1} - 1$. The last section considers the general case.

(¹) A. Emch: *On conformal Rational Transformations in a Plane*. Rendiconti del Circolo Matematico di Palermo, XXXIV (1912), pp. 1-12. On Stelloids in general see: G. Loria, *Spezielle Algebraische und Transcendente Ebene Kurven*, 15th Kap.—*Geometrie der Polynome*, Vol. I (1902) pp. 368-80; C. E. Brooks: *A Note on the Orthic Cubic Curve*, Johns Hopkins University Circular (1904), pp. 47-52, and *Orthic Curves, or Algebraic curves which satisfy Laplace's equation in two dimensions*, Proceedings of American Philosophical Society, Vol. XLIII (1904), pp. 294-331.

The transformation $z' = \frac{1}{z}$ is studied in detail in the article by Professor Emch, *Involutoric Circular Transformations as a Particular Case of the Steinerian Transformation and their Invariant Nets of Cubics*, Annals of Mathematics, 2nd Series, XIV (1912), pp. 57-71.

For some of the work, use will be made of the following

Theorem I. *The product of a pencil of curves and the second polar pencil of a point (x', y') is identical with the polar of the product of the pencil and the first polar pencil of (x', y') (¹).*

For, let the pencil of curves be

$$(1) \quad P + \lambda Q = 0,$$

and the first and second polar pencils

$$(2) \quad \Delta P + \lambda \Delta Q = 0, \text{ and}$$

$$(3) \quad \Delta^2 P + \lambda \Delta^2 Q = 0, \text{ respectively.}$$

The product of pencils (1) and (2) is

$$(4) \quad P \cdot \Delta Q - Q \cdot \Delta P = 0, \text{ whose polar is}$$

$$\Delta P \cdot \Delta Q + P \cdot \Delta^2 Q - \Delta Q \cdot \Delta P - Q \cdot \Delta^2 P = 0 \quad \text{or}$$

$$(5) \quad P \cdot \Delta^2 Q - Q \cdot \Delta^2 P = 0,$$

which is identical with the product(²) of (1) and (3).

II. Transformation $z' = z - \frac{3(z^3 - 1)}{3z^2} = \frac{1}{z^2}$.

Geometrically, this transformation represents an inversion, a reflexion, doubling of the angle and squaring of the absolute value. For it may be replaced by two transformations, $z'' = \frac{1}{z}$ and $z' = z''^2$, whose properties are well known. Straight lines are reflected on the x -axis and their inclinations are doubled. The unit circle corresponds to itself but only the three points $(1, 0)$ $\left(-\frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$, and $\left(-\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right)$ are invariant. The three lines joining these three points and the origin are also invariant lines but not point-wise. An equilateral hyperbola, $xy = c$, goes into the circle, $2c(x^2 + y^2) + y = 0$, counted twice. If (x', y') describes a straight line, the point (x, y) describes a locus of the fourth order, since

(¹) This curve is called "Panpolare" by Steiner, who first investigated in a purely synthetic manner some of its properties in general, *Journal für die reine und angewandte Mathematik*, Vol. XLVII, pp. 70-82.

(²) On products of projective pencils see

Clebsch, *Vorlesungen über Geometrie*, Vol. I (1876) p. 375.

Cremona, *Theorie der ebenen Kurven* (German by Curtze, 1835) Paragraph 50, f.f.

Sturm, *Die Lehre von den geometrischen Verwandtschaften*, Vol. I (1909) p. 249, f.f.

Ency. der Math. Wiss. III, 2, 3, p. 353, f.f.

the points corresponding to (x', y') ⁽¹⁾ are the base-points of the first polar pencil of (x', y') with respect to the pencil of cubics (stelloids) through the three cube roots of unity and their associates⁽²⁾. Since the two transformations $z'' = \frac{1}{z}$ and $z' = z''^{1/2}$ are conformal around all points except 0 and ∞ , the result of using both of them is conformal, i.e., finite singularities of curves are preserved in the transformation $z' = \frac{1}{z^2}$.

Infinite points, however, are transformed into singularities at the origin.

The pencil of cubics is $u + \lambda v = 0$ where u and v are the real and imaginary parts, respectively, of $z^3 - 1$, i.e.,

$$(1) \quad u + \lambda v = x^3 - 3xy^2 - 1 + \lambda(3x^2y - y^3) = 0.$$

The projective pencil of first polars is

$$(2) \quad (x^2 - y^2)x' - 2xyy' - 1 + \lambda[2xyx' + (x^2 - y^2)y'] = 0;$$

pencil of second polars is

$$(3) \quad (x'^2 - y'^2)x - 2x'y'y - 1 + \lambda[2x'y'x + (x'^2 - y'^2)y] = 0.$$

The product of (1) and (2) is, as we should expect from the general theory, a bicircular quintic

$$(4) \quad (x'y - y'x)[(x^2 + y^2)^2 + 2x] + (y' - y)(3x^2 - y^2) = 0.$$

The product of (1) and (3) is

$$(5) \quad 2(x^2 + y^2)(xx' + yy')(x'y - xy') + (x'^2 - y'^2)y + 2xx'y' - 3x^2y' - 3x^2y + y^3 = 0,$$

a circular quartic; the first polar of (x', y') with respect to (4), in agreement with Theorem I.

The product of (2) and (3) is the circular cubic

$$(6) \quad (x'^2 - y'^2)(x^2yx' + x'y^3 + x^3y' + xy'y^2 + y) + 2x'y'(x^2yy' + y'y^3 - xx'y^2 - x'x^3 + x) - 2xx'y - y'(x^2 - y^2) = 0.$$

This cubic belongs to the class discussed by Emch in the paper referred to on p. 1, and will not be studied here.

The Quintic (4).

Since there are no terms of the fourth degree in equation (4), and $(x'y - y'x)$ is a factor of the fifth degree terms, the line $x'y - y'x = 0$ is

⁽¹⁾ L. Cremona: *Theorie der ebenen Kurven* (German by Curtze, 1865) p. 120, Lehrsatz XI

⁽²⁾ A. Emch: (l.c. p. 1) pp. 8 and 12.

an asymptote. There is a double point at the origin and at each of the circular points, I and J , at infinity. The curve passes through the base-points of (1) and (2), viz., the points

$$(1, 0); \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right); \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right);$$

$$\left(\pm \frac{1}{2} \frac{\sqrt{x' - iy'} \pm \sqrt{x' + iy'}}{\sqrt{x'^2 + y'^2}}, \pm \frac{i}{2} \frac{\sqrt{x' + iy'} \mp \sqrt{x' - iy'}}{\sqrt{x'^2 + y'^2}}\right).$$

At the first three points above, $\frac{dy}{dx}$ has the values $\frac{y'}{x' - 1}$, $\frac{2y' - \sqrt{3}}{2x' + 1}$ and $\frac{2y' + \sqrt{3}}{2x' + 1}$, respectively. These show that the tangents at these

three points, which are the points representing the three cube roots of unity, pass through the pole (x', y') . This follows directly from the fact that (5), the first polar of (x', y') with respect to (4), passes through these three points. At the origin $\frac{dy}{dx} = \frac{x' \pm \sqrt{x'^2 + y'^2}}{y'}$, i.e., the tangents to the

curve at the origin are $y = \frac{x' \pm \sqrt{x'^2 + y'^2}}{y'} x$, which are orthogonal. If

θ is the inclination of either of these tangents, $\tan 2\theta = -\frac{y'}{x'}$. Hence,

to construct the tangents to (4) at the origin, join the origin to the point $(x', -y')$ and bisect the angles made by this line with the x -axis. The bisectors are the required tangents. These tangents form the only real degenerate conic of the pencil (2), and are obtained also by putting $\lambda = \infty$ in equation (2).

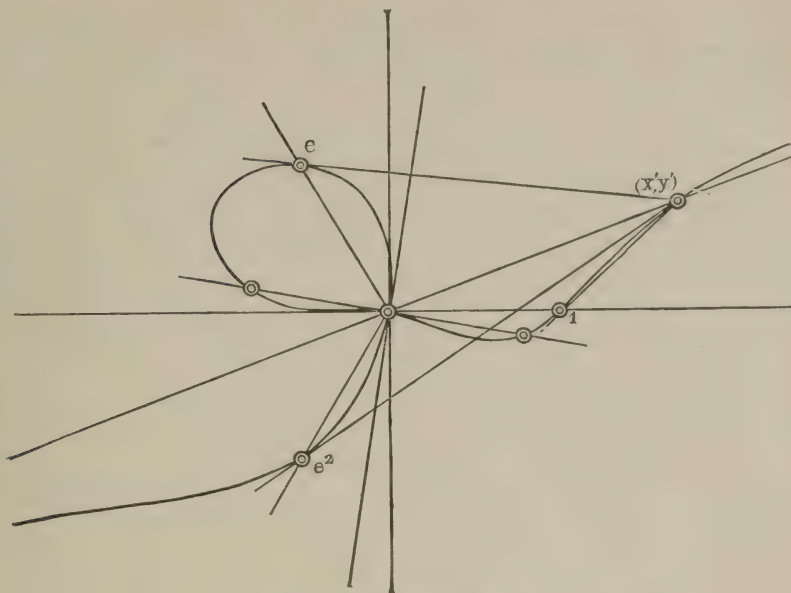
This is sufficient to enable us to make a fairly accurate drawing of the curve. (See figure on next page.)

Some of the properties of (4) appear more readily if it is put into the polar form

$$(7) \quad \rho^2 [\rho^3 (x' \sin \theta - y' \cos \theta) - \rho \sin 3\theta + x' \sin 2\theta + y' \cos 2\theta] = 0.$$

The factor ρ^2 indicates again, that the origin is a double point. If $x' \sin 2\theta + y' \cos 2\theta = 0$, or $\tan 2\theta = -\frac{y'}{x'}$, one value of ρ is zero. The others are obtained from

$$\rho^2 (x' \sin \theta - y' \cos \theta) = \sin 3\theta,$$



Graph of the quintic (4).

whence $\rho = \pm \frac{1}{\sqrt[4]{x'^2 + y'^2}}$ provided we consider $\sin 2\theta$ negative and $\cos 2\theta$ positive. An interchange of signs would make ρ imaginary. (The fourth root arises from the fact that the functions of θ involve the square root.) The curve cuts one of the tangents at the origin in two points equidistant from the origin. These two points are the real base-points of (2) as may be verified by making use of the coördinates of the base-points as given on p. 4. Since the coefficient of ρ^2 within the bracket is zero, the sum of the three non-vanishing segments on any ray through the origin vanishes. The origin is therefore a center of the curve.

More than this, the polar equation (7), gives us a hint as to the form of the equation of the product curve for $n > 2$. This will be discussed later.

Quadruple foci of (4).

Foci are sect-points of tangents from the circular points to a curve⁽¹⁾. The tangents at I and J are of the form $y = ix + b$ and $y = ix + c$, re-

(1) Bassett, *Elementary Treatise on Cubic and Quartic Curves*, p. 46.

Charlotte A. Scott, *Modern Analytical Geometry*, p. 122.

Numerous special cases of foci are treated by R. A. Roberts, *On Foci and Confocal Plane Curves*, *Quarterly Journal of Mathematics* XXXV (1903-4), pp. 297-384.

spectively. Putting $b = \beta + ia$, then (a, β) is the only real point on the tangent, i.e., it is the focus. Substituting $y = -ix + b$ in equation (4) we have

$$(8) \quad (4i + 4ib^2x' + 4b^2y')x^3 + (4ib^3y' - 6b - 2ix' - 8b^3x' + 2y')x^2 \\ + (2bx' + 2iby' - 3ib^2 - b^4y' - 5ib^4x')x + b^3 - b^2y' + b^5x' = 0.$$

The degree reduces to 3 because the circular points are double points. In order for $y = -ix + b$ to be tangent to I , the coefficient of x must also vanish, i.e.,

$$b^2 = -\frac{1}{x' - iy'}.$$

Hence the tangents at I are $y = -i\left(x \pm \frac{1}{\sqrt{x' - iy'}}\right)$. Similarly, the tangents at J are $y = i\left(x \pm \frac{1}{\sqrt{x' + iy'}}\right)$. These intersect in the four points (two of them real)

$$\left(\pm \frac{1}{2} \frac{\sqrt{x' - iy'} \pm \sqrt{x' + iy'}}{\sqrt{x'^2 + y'^2}}, \pm \frac{i}{2} \frac{\sqrt{x' + iy'} \mp \sqrt{x' - iy'}}{\sqrt{x'^2 + y'^2}}\right),$$

which are the base-points of the pencil (2). We have seen that the orthogonal tangents at the origin are the two lines of the real degenerate equilateral hyperbola of the pencil (2). Hence we may state the

Theorem II. *The three degenerate equilateral hyperbolas of the pencil (2) are the tangents to the curve (4) at the double points, which are their vertices. The base-points of the pencil (2) are foci of (4).*

Single foci of (4).

The quintic (4) has three double points and no other singularities. Its class is therefore $5(5-1) - 3 \cdot 2 = 14$. Since the circular points are double points we can draw from each of them only 10 tangents touching the curve elsewhere. The 100 intersections of these ten tangents are foci of the curve, but only 10 of these are real. They belong to the 196 base-points of a pencil of curves of order 14. Each of the 10 tangents from I cuts each of the two tangents at J in two coincident points (double foci), thus yielding 40 double foci; similarly, the tangents from J determine 40 double foci. The tangents at I and J determine four quadruple foci (2 real) considered above, counting for 16 points. Thus we have accounted for $100 + 40 + 40 + 16 = 196$ base-points. To determine

the real single foci, impose on equation (8) the condition that it shall have equal roots, i.e., that the discriminant shall vanish. To obtain the discriminant, take the derivative with respect to x and solve the quadratic so obtained for x . Since the double roots of the cubic (8) must also be roots of its derived equation, we reverse the process and substitute the roots of the derived equation in (8). The two expressions thus obtained are the two factors of the discriminant. These, set equal to zero, are $b - ix' - y' = 0$, and

$$(9) \quad 4b^3(x+iy)^3 + 27b^7(x-iy)^2 - b^6(15ix^3 + 117x^2y - 45ixy^2 - 39y^3) \\ + 54b^5(x-iy) - 54ib^4(x-iy)^2 - 12b^3(x+iy)^3 \\ + 27b^3 - 27ib^2(x-iy) - 4i(x+iy)^3 = 0.$$

Hence, the line $y = -ix + ix' + y'$ is a tangent to the curve (4). The real point on it is (x', y') , the pole, which is therefore a focus. (As is well known, the corresponding value of c is $y' - ix'$.) By equation (9), the other nine real single foci are so situated that the origin is their centroid and the product of their distances from the origin has an absolute value equal to unity. The former follows from the fact that the eighth degree term is missing; the latter is seen by dividing through by the coefficient of b , when the constant term reduces to i .

Since the inverse of a focus is the focus of the inverse curve, the problem of finding the foci of (4) reduces to that of finding the foci of its inverse with respect to the origin, viz., a circular quartic

$$(10) \quad (x^2 + y^2)[2x'xy + y'(x^2 - y^2)] + y^3 - 3x^2y + x'y - y'x = 0.$$

This does not simplify matters, however.

Isotropic Coordinates.

The problem of finding foci is much simpler when the equation of the curve is expressed in isotropic coordinates.⁽¹⁾ Put $z = x + iy$, $\bar{z} = x - iy$,

(1) A. Perna, *Le Equazioni delle Curve in Coordinate Complesse Coniugate*, Rendiconti del Circolo Matematico di Palermo XVII (1903) pp. 65-72.

Beltrami, *Ricerche sulla Geometria delle forme binarie cubiche*, Memorie dell'Acc. di Bologna X (1870) p. 626.

Cesàro, *Sur la détermination des foyers des coniques*, Nouvelles Annales des Mathématiques LX (1901) pp. 1-9.

G. Lery, *Sur la fonction de Green*, Annales Scientifiques de l'École Normale Supérieure, XXXII (1915) pp. 49-135.

C. E. Brooks, (l. c. p. 1.) calls these conjugate coordinates. Cayley (Collected Works VI, p. 498) uses the name circular coordinates.

or $x = \frac{z + \bar{z}}{2}$, $y = \frac{z - \bar{z}}{2i}$. Equation (4) becomes

$$(11) \quad f \equiv [(x' - iy')\bar{z}^2 - 1]z^3 - [(x' + iy')(\bar{z}^3 - 1)]z^2 - \bar{z}^2(x' - iy') + \bar{z}^3 = 0,$$

$$(12) \quad \frac{\partial f}{\partial z} = 3[(x' - iy')\bar{z}^3 - 1]z^2 - 2[(x' + iy')(\bar{z}^3 - 1)]z = 0.$$

The roots of (12) are $z = 0$, $z = \frac{2(x' + iy')(\bar{z}^3 - 1)}{3(x' - iy')\bar{z}^2 - 1}$. Substituting $z = 0$ in (11), we get $\bar{z} = x' - iy'$, i.e., x' , y' is a focus.⁽¹⁾ Substituting the second root of (12) in (11), we have

$$(13) \quad 4(x' + iy')^3(\bar{z}^3 - 1)^3 - 27\bar{z}^3[(x' - iy')\bar{z}^2 - 1]^2 + 27\bar{z}^2(x' - iy')[(x' - iy')\bar{z}^2 - 1]^2 = 0.$$

If in equation (13), z is replaced by its equivalent, $-ib$, the result is identical with equation (9), as it should be.

III. Transformation $z' = z - \frac{(n+1)(z^{n+1} - 1)}{(n+1)z^n} = \frac{1}{z^n}$.

The pencil of stelloids connected with this transformation is the pencil of curves through the $n+1$ th roots of unity and their associates. The transformation represents a $(1, n)$ correspondence between the pole (x', y') and the n real base-points of the first polar pencil of (x', y') . To establish the equation of the pencil and first polar in polar coordinates we have

$$u + iv = z^{n+1} - 1 = \rho^{n+1} \cos(n+1)\theta + i\rho^{n+1} \sin(n+1)\theta - 1 = 0.$$

The pencil of stelloids is

$$(14) \quad u + \lambda v = \rho^{n+1} \cos(n+1)\theta - 1 + \lambda \rho^{n+1} \sin(n+1)\theta = 0.$$

The first polar pencil of (x', y') is

$$(15) \quad u_1 + \lambda v_1 = \rho^n x' (n+1) \cos n\theta - y' \rho^n (n+1) \sin n\theta - (n+1) + (n+1)\lambda [x' \rho^n \sin n\theta + y' \rho^n \cos n\theta] = 0.$$

The product of (14) and (15) is

$$(16) \quad \rho^n \{ \rho^{n+1} (x' \sin \theta - y' \cos \theta) - \rho \sin(n+1)\theta + x' \sin n\theta + y' \cos n\theta \} = 0,$$

which may be written in the form

$$(17) \quad \rho^{2n} (x' y - y' x) - \rho^{n+1} \sin(n+1)\theta + x' \rho^n \sin n\theta + y' \rho^n \cos n\theta = 0.$$

(1) Lery, (l. c. p. 7) p. 51. Brooks (l. c. p. 1) p. 309.

This shows that in cartesian coordinates, $(x^2 + y^2)^n$ is a factor of the terms containing x and y to degree $2n+1$ while the next highest power of x and y is $n+1$. Hence the

Theorem III. *The product of the pencil of stelloids determined by the $n+1^{\text{th}}$ roots of unity and their associates as base-points and the first polar pencil of a point (x', y') , is a circular curve (16) having an n -fold point at each of the circular points and at the origin.*

Also since $x'y - y'x$ is a factor of the highest degree terms, and the terms of order $n+2$ to $2n$ are missing, we have

Theorem IV. *The line $x'y - y'x = 0$ joining the origin and the pole is an asymptote of the product (16), if $n > 1$. The sum of segments on rays through the origin is zero, i.e., the origin is a center.*

If in the second factor of (16) we put $\rho = 0$, we get

$$x' \sin n\theta + y' \cos n\theta = 0,$$

that is

Theorem V. *The tangents to (16) at the origin are the lines $y = x \tan \theta$, where $\tan n\theta = -\frac{y'}{x'}$. These n tangents divide the whole angle about the origin into n equal parts, beginning at $y = x \tan \phi$, where $\phi = \arctan \left(-\frac{y'}{x'} \right) = n\theta + 2k\pi$.*

Making use of the values $\theta = \frac{\phi}{n} - \frac{2k\pi}{n}$ ($k=0, 1, 2, \dots, (n-1)$) we find that the curves cuts the tangents at the origin in the points $\rho^n (x' \sin \theta - y' \cos \theta) = \sin(n+1)\theta$, or $\rho^n = \pm \frac{1}{\sqrt{x'^2 + y'^2}}$, according as $\cos n\theta$ or $\sin n\theta$ is considered positive, i.e., the curve cuts in other real points all of these tangents if n is odd and cuts only half of them elsewhere if n is even. The points of intersection are the base-points of (15), as may be easily verified by substituting their coordinates in (15).

The tangents at the origin constitute the degenerate curve obtained by making $\lambda = \infty$ in (15). A general theorem⁽¹⁾ states that if two corresponding curves C^m and C^n in two projective pencils of curves have a common multiple point of multiplicities r and s ($r < s$) respectively, their product K has there a multiple point of order r and the r tangents of K are tangents to C^m . We have here an example in which both C^m and C^n are the real degenerate members of the two pencils. In fact, each

(1) Sturm, (l. c. p. 2): Ency. der Math. Wiss. III 2, 3, p. 355.

of them consists of straight lines through the origin, the C^n being the $n+1$ straight lines through the origin obtained by making $\lambda=\infty$ in (14).

Single foci of (16).

Introducing isotropic coordinates

$$z = x + iy = \rho(\cos \theta + i \sin \theta); \quad \bar{z} = x - iy = \rho(\cos \theta - i \sin \theta)$$

in equation (16), it reduces to

$$(18) \quad f \equiv [(x' - iy')\bar{z}^n - 1]z^{n+1} - [(x' + iy')(\bar{z}^{n+1} - 1)]z^n + \bar{z}^{n+1} - (x' - iy')\bar{z}^n = 0.$$

To find the foci, impose the condition on (23) that it shall have equal roots in $z^{(1)}$. To do this, we get

$$(19) \quad \frac{\partial f}{\partial z} = (n+1)[(x' - iy')\bar{z}^n - 1]z^n - n[(x' + iy')(\bar{z}^{n+1} - 1)]z^{n-1} = 0.$$

If a root of (19) is also a root of (18), it is a double root of (18). Equation (19) has $(n-1)$ roots $z=0$. In order for $z=0$ to be a root of (18), we must have $\bar{z}^n[\bar{z} - (x' - iy')] = 0$, i.e., $\bar{z} = x' - iy'$, whence the pole (x', y') is a focus. $\bar{z}=0$ signifies merely that the origin is a multiple point. The remaining root of (19) is $z = \frac{n[(x' + iy')(\bar{z}^{n+1} - 1)]}{(n+1)[(x' - iy')\bar{z}^n - 1]}$. To find the condition that this shall be a root of (18) it is substituted in (18) giving the condition

$$(20) \quad n^n [(x' + iy')(\bar{z}^{n+1} - 1)]^{n+1} - (n+1)^{n+1} \bar{z}^n [\bar{z} - (x' - iy')][(x' - iy')\bar{z}^n - 1]^n = 0.$$

The highest power of \bar{z} in this equation is $(n+1)^2$ and the next highest power is $n^2 + n + 1 = (n+1)^2 - n$. Hence, for $n > 1$, the coefficient of the next highest power of \bar{z} vanishes and the origin is the centroid of the roots of (20), i.e., of the single foci of (18), or (16). Also the constant term of (20) arises in the first bracket and has the same coefficient, except for sign, as the highest power of \bar{z} , i.e., the product of the roots of (20) is ± 1 , according as n is even or odd.

If in (18) we set the coefficient of z^{n+1} equal to zero, we get at once the double foci, viz., the points $\bar{z} = \frac{1}{\sqrt[n]{x' - iy'}}$, which are the base-points of (15). Hence,

Theorem VI. *The base-points of the first polar pencil (15) are foci of (16).*

(¹) See Lery or Brooks (l. c. p. 8).

First polar of (16).

The product of (14) and the second polar pencil of (x', y') is the first polar of (16), viz.,

$$(21) \quad \rho^{n-1} \{ \rho^{n+1} [(x'^2 - y'^2) \sin 2\theta - 2x'y' \cos 2\theta] - \rho^2 \sin (n+1)\theta + (x'^2 - y'^2) \sin (n-1)\theta + 2x'y' \cos (n-1)\theta \} = 0.$$

Since the difference in degree of the two highest power of ρ is $n-1$, for $n > 2$, the asymptotes are determined by

$$(22) \quad (x'^2 - y'^2) \sin 2\theta - 2x'y' \cos 2\theta = 0.$$

From this, $\tan 2\theta = \frac{2m}{1-m^2}$ where $m = \frac{y'}{x'}$. Moreover, since $\tan 2\theta = \tan 2\left(\theta + \frac{\pi}{2}\right)$, it follows that these are the lines joining (x', y') to the origin and the line normal to it at the origin.

The tangents at the origin are determined by

$$(23) \quad (x'^2 - y'^2) \sin [(n-1)\theta + 2k\pi] + 2x'y' \cos [(n-1)\theta + 2k\pi] = 0,$$

since in (21) this is the condition for a root $\rho = 0$. From (23) we get $\tan (n-1)\theta = -\frac{2x'y'}{x'^2 - y'^2} = \tan 2\left[\arctan\left(-\frac{y'}{x'}\right)\right]$, or $(n-1)\theta = 2\varphi + m\pi = -2A + m\pi$, where A is the inclination of the line joining the origin and the pole (x', y') . For $\sin (n+1)\theta = 0$, $\rho^{n+1} = \cos (n+1)\theta$, or $\rho = 1$. Hence the curve (21) passes through the $(n+1)$ th roots of unity. But the curve (16) with respect to which (21) is the first polar of (x', y') , also passes through these points. We have therefore the

Theorem VII. *The lines joining the pole (x', y') to the $(n+1)$ th roots of unity are tangents to the curve (16).*

IV. The general transformation $z' = z - \frac{(n+1)f(z)}{f'(z)}$.

In the general case⁽¹⁾ $f(z) = u + iv = a_0 \prod_1^{n+1} ((z - z_k) = 0$. This may be thought of as representing $n+1$ lines⁽²⁾ through the circular point I .

(¹) A. Emch (l. c. p. 1): p. 2.

(²) Compare C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici*, Math. Annalen XL (1892), pp. 413-467.

The pencil of stelloids is $u + \lambda v = 0$ and the $n+1$ lines are determined by the value $\lambda = i$.

Similarly, for $\lambda = i$, the first polar pencil $u_1 + \lambda v_1 = 0$ represents n lines through I and the base-points of the first polar pencil. Also $u - iv = 0$ and $u_1 - iv_1 = 0$ represent sets of lines through J .

By the general theorem regarding multiplicities of products, p. 2, we then have (assuming that the general theorem applies to imaginary elements)

Theorem VIII. *The product $uv_1 - u_1v = 0$ of the projective pencils $u + \lambda v$ and $u_1 + \lambda v_1$ has an n -fold point at each of the circular points and the n lines $u_1 + iv_1 = 0$ are tangents to the product at I and the n lines $u_1 - iv_1$ are tangents to J .*

Since the sect-points of $u_1 + iv_1 = 0$ and $u_1 - iv_1 = 0$ are the base-points of the pencil $u_1 + \lambda v_1 = 0$, and these lines are tangents at I and J , we have

Theorem IX. *The n^2 base-points of the first polar pencil $u_1 + \lambda v_1 = 0$ are quadruple foci of the product $uv_1 - u_1v = 0$. Among these are the n real base-points forming n real foci.*

Since in the special cases treated the pole is a focus, we might expect that the pole is also, in general, a focus. This, however, is not the case.

Equation (27), p. 10 of the article by Emch referred to above is the equation of the product in general, viz.,

$$(24) \quad (x - x')(\dot{r}v - su) - (y - y')(ru + sv) = 0,$$

where r and s are $\frac{u'x}{n+1}$ and $\frac{v'x}{n+1}$ respectively. The form of (24) gives us the

Theorem X. *The product curve is also the product of the pencil of lines $(x - x') - \lambda(y - y') = 0$, through the pole, and the pencil of circular curves $(ru + sv) - \lambda(rv - su) = 0$.*

The line $x + iy = x' + iy'$, joining the pole and the circular point I , meets this curve in points of

$$(25) \quad (ru + sv) + i(rv - su) = 0,$$

which is identical with the expression just above equation (4), p. 4 of that article, where it is shown to be equal to

$$(26) \quad (a_0 + ib_0) \prod_1^{n+1} (x + iy - z_i) \prod_1^n (x - iy - \bar{z}_k) = 0.$$

Substituting the value of $x=x'+iy'-iy$, the first two sets of factors become constants and the third one gives n values of y which are

$$(27) \quad y = \frac{x' + iy' - \bar{z}_k}{2i} \quad (k=1, 2, \dots, n).$$

Hence two values of y cannot be equal unless two points of \bar{z}_k coincide. In the special cases treated, $\bar{z}_k=0$ so that the pole is a focus, except for the case $n=1$, which gives only one value of y . In general, two values of y cannot be equal in (27) and the pole is not a focus.

This follows directly for general positions of the pole, since in this case (25) is independent of (x', y') .

From the equation of the tangent line and equation (27) the point of contact is $\left(\frac{x'+iy'}{2}, \frac{x'+iy'}{2i}\right)$ for the case $\bar{z}_k=0$. (The tangent has contact of order $n-1$ at this point.) This point lies on the line $y=-ix$. In the same way it may be shown that the point of contact of the tangent joining the pole to the circular point J lies on the line $y=ix$. Hence

Theorem XI. *In the special cases of sections II and III, the circular points, the pole, the origin, and the two points of contact of the tangents joining the pole to the circular points, are the vertices of a complete quadrilateral, i.e., the points of contact of these tangents are the associate points of the pole and origin.*

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Über Variationsrechnung im Raume,

von

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Es sei ein Bereich des Raumes gegeben und zu jedem Linienelement, das im Innern von R liegt und durch seine Koordinaten x, y, z und die Richtungscosinus $\cos \theta, \cos \varphi, \cos \psi$ mit den Axen Ox, Oy, Oz charakterisiert ist, eine positive Funktion $V(x, y, z; \theta, \varphi, \psi)$ zugeordnet, welche der Einfachheit halber, vorausgesetzt dass es eine analytische Funktion der Variablen x, y, z und periodisch der θ, φ, ψ mit der Periode 2π ist. Es sei andererseits ein Bogen P_1P_2 der rektifizierbaren Kurve, welcher durch die Gleichungen:

$$(1) \quad x=x(t), \quad y=y(t), \quad z=z(t), \quad t_1 \leq t \leq t_2$$

dargestellt ist, der im Innern von R liegt und die zwei gegebenen Punkte $P_1(t_1), P_2(t_2)$ des Bereiches verbindet. Das Integral:

$$(2) \quad I = \int_{t_1}^{t_2} \frac{\sqrt{x'^2 + y'^2 + z'^2}}{V(x, y, z; \theta, \varphi, \psi)} dt,$$

$$\theta = \arccos \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}}, \quad \varphi = \arccos \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}},$$

$$\psi = \arccos \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}},$$

wird einen vollständig definierten Wert haben.

Betrachtet man nun die Funktion V als die dem Linienelement $(x, y, z; \theta, \varphi, \psi)$ entsprechende Geschwindigkeit für einen Punkt, der sich auf diese Kurve bewegt, so stellt das Integral (2) die Zeit dar, welche der bewegte Punkt braucht, wenn er den Bogen P_1P_2 durchläuft. Verlangt man nun die Kurve (1) im R so zu bestimmen, dass diese Zeit die kleinste sei, so hat man das Problem der Variationsrechnung im Raume; setzt man ferner:

$$(3) \quad \frac{\sqrt{x'^2 + y'^2 + z'^2}}{V(x, y, z; \theta, \varphi, \psi)} = F(x, y, z; x', y', z'),$$

so sieht man nach der Definition selbst, dass die Funktion F die Homogenitätsrelation erfüllt:

$$(4) \quad F(x, y, z; kx', ky', kz') = kF(x, y, z; x', y', z'), \quad k > 0,$$

für alle Werte von x, y, z im R und von x', y', z' die nicht zu gleicher Zeit gleich Null sind.

Auf Grund der Relation (3) nimmt das Integral (2) folgende Form an:

$$(2') \quad I = \int_{t_1}^{t_2} F(x, y, z; x', y', z') dt.$$

Betrachtet man nun die Bogenlänge s , von einem bestimmten Punkt der Kurve (1) gemessen, als unabhängigen Parameter, so hat man:

$$x' = \frac{dx}{ds} = \cos \theta, \quad y' = \frac{dy}{ds} = \cos \varphi, \quad z' = \frac{dz}{ds} = \cos \psi,$$

wobei θ, φ, ψ die Winkel der positiven Richtung der Tangente der Kurve (1) im Punkte $P(x, y, z)$ mit den Axen bedeuten. Setzt man weiter zur Abkürzung:

$$F(x, y, z; x', y', z') = F(x, y, z; \theta, \varphi, \psi) = F(\theta, \varphi, \psi),$$

so ergibt sich wegen der Homogeneitätseigenschaft, dass:

$$(5) \quad F(\theta, \varphi, \psi) = \cos \theta F_x(\theta, \varphi, \psi) + \cos \varphi F_y(\theta, \varphi, \psi) + \cos \psi F_z(\theta, \varphi, \psi),$$

$$(6) \quad \begin{cases} F_x(\theta, \varphi, \psi) = \cos \theta F_{x'x}(\theta, \varphi, \psi) + \cos \varphi F_{y'x}(\theta, \varphi, \psi) + \cos \psi F_{z'x}(\theta, \varphi, \psi), \\ F_y(\theta, \varphi, \psi) = \cos \theta F_{x' y}(\theta, \varphi, \psi) + \cos \varphi F_{y' y}(\theta, \varphi, \psi) + \cos \psi F_{z' y}(\theta, \varphi, \psi), \\ F_z(\theta, \varphi, \psi) = \cos \theta F_{x' z}(\theta, \varphi, \psi) + \cos \varphi F_{y' z}(\theta, \varphi, \psi) + \cos \psi F_{z' z}(\theta, \varphi, \psi), \end{cases}$$

$$(6') \quad \begin{cases} \cos \theta F_{x'x}(\theta, \varphi, \psi) + \cos \varphi F_{x'y'}(\theta, \varphi, \psi) + \cos \psi F_{x'z'}(\theta, \varphi, \psi) = 0, \\ \cos \theta F_{y'x'}(\theta, \varphi, \psi) + \cos \varphi F_{y'y'}(\theta, \varphi, \psi) + \cos \psi F_{y'z'}(\theta, \varphi, \psi) = 0, \\ \cos \theta F_{z'x'}(\theta, \varphi, \psi) + \cos \varphi F_{z'y'}(\theta, \varphi, \psi) + \cos \psi F_{z'z'}(\theta, \varphi, \psi) = 0, \end{cases}$$

und nach einer Differentiation nach x, y, z, x', y', z' der (4) findet man:

$$F_x(x, y, z; kx', ky', kz') = kF_x(x, y, z; x', y', z'),$$

$$\dots\dots\dots$$

$$F_{x'}(x, y, z; kx', ky', kz') = F_{x'}(x, y, z; x', y', z'),$$

$$\dots\dots\dots$$

Aus den Gleichungen (6') findet man leicht, dass:

$$(7) \quad \begin{cases} \frac{F_{x'x'}F_{y'y'} - F_{x'y'}^2}{\cos^2 \psi} = \frac{F_{x'x'}F_{z'z'} - F_{x'z'}^2}{\cos^2 \varphi} = \frac{F_{y'y'}F_{z'z'} - F_{y'z'}^2}{\cos^2 \theta}, \\ \frac{F_{x'y'}F_{z'z'} - F_{x'z'}F_{y'z'}}{-\cos \theta \cos \varphi} = \frac{F_{x'z'}F_{y'y'} - F_{x'y'}F_{y'z'}}{-\cos \theta \cos \psi} = \frac{F_{y'z'}F_{x'x'} - F_{x'x'}F_{x'y'}}{-\cos \varphi \cos \psi} \\ = F_1(x, y, z; \theta, \varphi, \psi). \end{cases}$$

Ist nun der Bogen (1), wobei s statt t eingesetzt wurde, ein solcher, der das Integral (2') zu einem Minimum macht, so findet man folgende Gleichungen:

$$F_{x'}(\theta, \varphi, \psi) = \int_{s_1}^{s_2} F_x(\theta, \varphi, \psi) ds + c_1,$$

$$F_{y'}(\theta, \varphi, \psi) = \int_{s_1}^{s_2} F_y(\theta, \varphi, \psi) ds + c_2,$$

$$F_{z'}(\theta, \varphi, \psi) = \int_{s_1}^{s_2} F_z(\theta, \varphi, \psi) ds + c_3,$$

aus denen man die folgenden Lagrangeschen Differentialgleichungen des betrachteten Problems erhält⁽¹⁾:

$$(8) \quad \begin{cases} \frac{d}{ds} F_{x'}(\theta, \varphi, \psi) - F_x(\theta, \varphi, \psi) = 0, \\ \frac{d}{ds} F_{y'}(\theta, \varphi, \psi) - F_y(\theta, \varphi, \psi) = 0, \\ \frac{d}{ds} F_{z'}(\theta, \varphi, \psi) - F_z(\theta, \varphi, \psi) = 0. \end{cases}$$

Durch eine Beweisführung, welches analog dem entsprechenden Ebene-Variationsproblem ist, findet man leicht die Weierstrasssche notwendige Bedingung für ein Minimum, d. h. dass:

$$(9) \quad \mathcal{E}(\theta, \varphi, \psi; \bar{\theta}, \bar{\varphi}, \bar{\psi}) \geq 0;$$

wobei: $\mathcal{E}(\theta, \varphi, \psi; \bar{\theta}, \bar{\varphi}, \bar{\psi})$

$$= \mathcal{E}\{x(s), y(s), z(s); x'(s), y'(s), z'(s); \cos \bar{\theta}, \cos \bar{\varphi}, \cos \bar{\psi}\}$$

$$\mathcal{E} = (\bar{F}_{x'} - F_{x'})\bar{x}' + (\bar{F}_{y'} - F_{y'})\bar{y}' + (\bar{F}_{z'} - F_{z'})\bar{z}'$$

ist und $\bar{\theta}, \bar{\varphi}, \bar{\psi}$ diejenigen Winkel bedeuten, welche die positive Richtung der Tangente einer Kurve C_1 im Schnittpunkte derselben mit der Kurve (1) bilden. Diese letzte Grösse kann man unter der Form schreiben:

$$(10) \quad \frac{1}{2} [F_{x'x'}\xi^2 + F_{y'y'}\eta^2 + F_{z'z'}\zeta^2 + 2F_{x'y'}\xi\eta + 2F_{x'z'}\xi\zeta + 2F_{y'z'}\eta\zeta],$$

wobei:

$$\bar{F} = \bar{F}(x, y, z; \bar{x}', \bar{y}', \bar{z}'), \quad \bar{F}_{x'} = F_{x'}(x, y, z; \bar{x}', \bar{y}', \bar{z}'),$$

$$F_{x'x'} = F_{x'x'}(x, y, z; x' + \theta\xi, y' + \theta\eta, z' + \theta\zeta), \quad \text{u.s.w.,} \quad 0 < \theta < 1$$

$$\bar{x}' - x' = \xi, \quad \bar{y}' - y' = \eta, \quad \bar{z}' - z' = \zeta$$

(1) Vergl. M. Mason and G. Bliss: *The properties of curves in spaces which minimize a definite integral*: Transactions of the American Mathematical Society 9, 1908, p. 440.

gesetzt ist. Aus der Bedingung (9) schliesst man, dass im Falle eines Minimums die quadratische Form (10) positiv oder gleich Null sein muss für alle Wertsysteme der $x, y, z; x', y', z'$, welche der (1) entsprechen, und für alle Werte der ξ, η, ζ . Bezeichnen wir mit Q die quadratische Form (10), so haben wir:

$$2Q = F_{x'x'}\xi^2 + F_{y'y'}\eta^2 + F_{z'z'}\zeta^2 + 2F_{x'y'}\xi\eta + 2F_{x'z'}\xi\zeta + 2F_{y'z'}\eta\zeta.$$

Wenn man nun die Kroneckersche Transformation benutzt, so nimmt die Hessesche Determinante folgende Form an:

$$\begin{vmatrix} F_{x'x'} & F_{x'y'} & F_{x'z'} \\ F_{y'x'} & F_{y'y'} & F_{y'z'} \\ F_{z'x'} & F_{z'y'} & F_{z'z'} \end{vmatrix}$$

welche wegen (6') gleich Null ist; und wenn man die Funktion F_1 verschieden von Null voraussetzt, so kann man das folgende System

$$\begin{aligned} F_{x'x'}\xi + F_{x'y'}\eta + F_{x'z'}\zeta &= 0, \\ F_{y'x'}\xi + F_{y'y'}\eta + F_{y'z'}\zeta &= 0 \end{aligned}$$

nach ξ, η auflösen für jeden willkürlichen Wert von ζ . Wenn ξ_1, η_1 ein Wertsystem von ξ, η bedeuten, so haben wir:

$$(11) \quad 2Q \equiv F_{x'x'}(\xi - \xi_1)^2 + 2F_{x'y'}(\xi - \xi_1)(\eta - \eta_1) + F_{y'y'}(\eta - \eta_1)^2$$

und wir können diese letzte Grösse unter der Form einer Summe von zwei Quadraten setzen. Die Grösse (11) ist positiv, wenn $F_1 > 0$ und $F_{x'x'} > 0$ ist, woraus auch folgt, dass $F_{y'y'} > 0$ und ähnlicherweise $F_{z'z'} > 0$; dieselbe Grösse ist negativ, wenn $F_1 > 0$ und $F_{x'x'}, F_{y'y'}, F_{z'z'} < 0$ ist. Im Falle wo $F_1 = 0$ und $F_{x'x'} \neq 0$ ist, lässt sich diese Grösse unter der Form eines Quadrats schreiben, woraus sich ergibt, dass die Ableitungen erster Ordnung von $2Q$ nach ξ, η proportionell der entsprechenden Ableitung nach ζ sind. Wir haben also folgendes Resultat:

Die Lagrangeschen Bedingungen für ein Extremum des integrals I sind:

$$F_1 > 0, \quad F_{x'x'}, F_{y'y'}, F_{z'z'} > 0 \quad (\text{Minim.})$$

$$F_1 > 0, \quad F_{x'x'}, F_{y'y'}, F_{z'z'} < 0 \quad (\text{Maxim.})$$

Um nun diesen Bedingungen einen geometrische Erklärung zu geben, betrachten wir das Problem unter der ursprünglichen Form und ziehen im Raume den Hodograph der Geschwindigkeiten V , d. h. den Ort der Endpunkte aller Vektoren, welche durch einen Punkt des Raumes gehen und parallel der Geschwindigkeiten V sind, die einem bestimmten Punkt

x, y, z entsprechen. Dieser Ort ist eine geschlossene Fläche ($F > 0$), welche sich mit dem Punkt x, y, z ändert, und die wir „die Figurativ“ des räumlichen Problems nennen. Wenn wir mit λ, μ, ν die rechtwinkligen Koordinaten eines Punktes der Figurativ bezeichnen, so haben wir der Definition zufolge:

$$(12) \quad \begin{cases} \lambda = V \cos \theta = \frac{\cos \theta}{F(\theta, \varphi, \psi)}, \\ \mu = V \cos \varphi = \frac{\cos \varphi}{F(\theta, \varphi, \psi)}, \\ \nu = V \cos \psi = \frac{\cos \psi}{F(\theta, \varphi, \psi)}, \end{cases}$$

und folglich hat die Gleichung der Figurativ folgende Form:

$$(12') \quad F(x, y, z; \lambda, \mu, \nu) = \frac{\sqrt{\lambda^2 + \mu^2 + \nu^2}}{V} = 1.$$

Es seien $Q_1(\lambda, \mu, \nu)$, $\bar{Q}_1(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ zwei Punkte der Figurativ, welche den Richtungen (θ, φ, ψ) , $(\bar{\theta}, \bar{\varphi}, \bar{\psi})$ des Punktes (x, y, z) von (1) entsprechen; die Koordinaten des Punktes \bar{Q}_1 sind:

$$(12'') \quad \bar{\lambda} = \frac{\cos \bar{\theta}}{F(\bar{\theta}, \bar{\varphi}, \bar{\psi})}, \quad \bar{\mu} = \frac{\cos \bar{\varphi}}{F(\bar{\theta}, \bar{\varphi}, \bar{\psi})}, \quad \bar{\nu} = \frac{\cos \bar{\psi}}{F(\bar{\theta}, \bar{\varphi}, \bar{\psi})}.$$

Die Tangentialebene der Fläche (12') im Punkte Q_1 hat die Gleichung:

$$(13) \quad (X - \lambda)F_{x'}(x, y, z; \lambda, \mu, \nu) + (Y - \mu)F_{y'} + (Z - \nu)F_{z'} = 0,$$

wobei X, Y, Z die laufenden Koordinaten bedeuten.

Auf Grund der Relation (4) haben wir (5) und:

$$(13') \quad \lambda F_{x'} + \mu F_{y'} + \nu F_{z'} = F$$

und die Gleichung (13) nimmt folgende Form an:

$$XF_{x'} + YF_{y'} + ZF_{z'} = 1.$$

Aus dieser Gleichung sieht man, dass keine Tangentialebene der Figurativ durch den Anfangspunkt der Koordinaten geht, und wenn es zu einem Punkt der (1) zwei Richtungen (θ, φ, ψ) , $(\bar{\theta}, \bar{\varphi}, \bar{\psi})$ gibt, für welche die Bedingungen

$$(14) \quad \begin{cases} F_{x'}(\theta, \varphi, \psi) = F_{x'}(\bar{\theta}, \bar{\varphi}, \bar{\psi}), \\ F_{y'}(\theta, \varphi, \psi) = F_{y'}(\bar{\theta}, \bar{\varphi}, \bar{\psi}), \\ F_{z'}(\theta, \varphi, \psi) = F_{z'}(\bar{\theta}, \bar{\varphi}, \bar{\psi}) \end{cases}$$

erfüllt sind, und die analog den Erdmann-Weierstrassschen Bedingungen des diskontinuierlichen Ebeneproblems sind, so stimmen die Tangentialebenen der Figurativ in den Punkten Q_1 , \bar{Q}_1 welche diesen Richtungen entsprechen, überein.

Nun denken wir uns die Tangentialebenen der Figurativ im Punkte $Q_1(\lambda, \mu, \nu)$ und die Strecke $\bar{Q}_1\bar{L}_1$, welche vom Punkt \bar{Q}_1 parallel mit der O_1Q_1 bis zu der Ebene gezogen ist, wobei O_1 den Anfangspunkt der Koordinaten λ, μ, ν bedeutet. Es ist leicht zu sehen, dass folgende Relation besteht

$$\frac{\bar{Q}_1\bar{L}_1}{O_1Q_1} = \frac{\bar{Q}_1\bar{Q}'_1}{O_1Q'_1} = 1 - \bar{\lambda}F_{x'} - \bar{\mu}F_{y'} - \bar{\nu}F_{z'},$$

wenn man mit $O_1Q'_1$, $\bar{Q}_1\bar{Q}'_1$ die Abstände der Tangentialebene von den Punkten O_1 und \bar{Q}_1 bezeichnet. Auf Grund der (12'') kann man schreiben:

$$\begin{aligned} \frac{\bar{Q}_1\bar{L}_1}{O_1Q_1} &= \frac{F(\bar{\theta}, \bar{\varphi}, \bar{\psi}) - \cos \bar{\theta} F_{x'}(\theta, \varphi, \psi) - \cos \bar{\varphi} F_{y'}(\theta, \varphi, \psi) - \cos \bar{\psi} F_{z'}(\theta, \varphi, \psi)}{F(\bar{\theta}, \bar{\varphi}, \bar{\psi})} \\ &= \frac{\mathcal{E}(x, y, z; \theta, \varphi, \psi; \bar{\theta}, \bar{\varphi}, \bar{\psi})}{F(\bar{\theta}, \bar{\varphi}, \bar{\psi})}. \end{aligned}$$

Aus dieser letzten Relation sieht man, dass die Weierstrasssche Funktion \mathcal{E} ihr Vorzeichen ändert, wenn es Punkte der Figurativ gibt, welche auf beiden Seiten der Tangentialebene liegen. Die Weierstrasssche Bedingung ($\mathcal{E} \geq 0$) lässt sich in dieser geometrischen Interpretation ausdrücken, in dem man sagt, dass die Tangentialebene der Figurativ eine Extremale sein muss, wie die Tangente der Figurativ im entsprechenden Ebeneproblem⁽¹⁾. Man sieht auch, dass im Falle, wo die Bedingungen (14) erfüllt sind

$$\mathcal{E}(x, y, z; \theta, \varphi, \psi; \bar{\theta}, \bar{\varphi}, \bar{\psi}) = 0, \quad \bar{\mathcal{E}} = \mathcal{E}(x, y, z; \bar{\theta}, \bar{\varphi}, \bar{\psi}; \theta, \varphi, \psi) = 0$$

sind und wegen der Homogeneitätseigenschaft haben wir auch

$$\mathcal{E}(x, y, z; \lambda, \mu, \nu; \bar{\lambda}, \bar{\mu}, \bar{\nu}) = 0, \quad \mathcal{E}(x, y, z; \bar{\lambda}, \bar{\mu}, \bar{\nu}; \lambda, \mu, \nu) = 0,$$

welche in der geometrischen Sprache sagen, dass die Tangentialebene der Figurativ im Punkte $Q_1(\lambda, \mu, \nu)$ durch den Punkt $\bar{Q}_1(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ geht, und umgekehrt.

Betrachten wir nun ν als Funktion der zwei anderen Variablen λ, μ durch die Gleichung (12') definiert, und setzen

$$p = \frac{\partial \nu}{\partial \lambda}, \quad q = \frac{\partial \nu}{\partial \mu}, \quad r = \frac{\partial^2 \nu}{\partial \lambda^2}, \quad s = \frac{\partial^2 \nu}{\partial \lambda \partial \mu}, \quad t = \frac{\partial^2 \nu}{\partial \mu^2},$$

(¹) Vergl. C. Carathéodory: *Sur les points singuliers du problème du Calcul des Variation dans le plan*: Annali di Matematica pura ed applicata, (3) 21, 1913, p. 153.

so ist die totale Krümmung der Figurativ im Punkte Q_1 durch die Formel

$$\frac{rt - s^2}{(1 + p^2 + q^2)^2}$$

gegeben, und wenn man die p, q, r, s, t ersetzt, welche durch die Gleichungen:

$$\begin{aligned} F_{x'} + p F_{z'} &= 0, & F_{y'} + q F_{z'} &= 0, \\ F_{x'x'} + 2p F_{x'z'} + p^2 F_{z'z'} + r F_{z'} &= 0, \\ F_{x'y'} + q F_{x'z'} + p F_{y'z'} + p q F_{z'z'} + s F_{z'} &= 0, \\ F_{y'y'} + 2q F_{y'z'} + q^2 F_{z'z'} + t F_{z'} &= 0 \end{aligned}$$

bestimmt werden, so findet man mit Berücksichtigung der (13') und (12')

$$F_1(x, y, z; \lambda, \mu, \nu) \cdot \frac{(\lambda F_{x'} + \mu F_{y'} + \nu F_{z'})^2}{(F_{x'}^2 + F_{y'}^2 + F_{z'}^2)^2} = \frac{F_1}{(F_{x'}^2 + F_{y'}^2 + F_{z'}^2)^2}.$$

Daraus ergibt sich, dass, wenn $F_1(x, y, z; \theta, \varphi, \psi) > 0$ ist, die totale Krümmung der Figurativ, welche dem Punkte (x, y, z) der (1) entspricht, positiv ist.

Konstruktion einer Schar von gebrochenen Extremalen.

Wenn wir voraussetzen, dass man eine Umgebung des Punktes P_0 in R abgrenzen kann, so dass in jedem Punkt dieser Umgebung ein gebrochene Extremale, und nur eine, existiert benachbart der gebrochenen $P_1 P_0 P_2$, d. h. eine solche, die sich stetig an die ursprüngliche transformieren lässt, wenn die Punkt P sich auf einer Fläche, welche durch P_0 geht, bewegt, so können wir eine Schar von gebrochenen Extremalen konstruieren.

Es sei in der Tat eine Schar

$$(15) \quad x = \chi(s, a, b), \quad y = \psi(s, a, b), \quad z = \omega(s, a, b)$$

von Extremalen, welche von zwei Parametern a, b abhängig sind und den Bogen $P_1 P_0$ für $a = a_0, b = b_0$ enthalten und die Eigenschaft besitzen, dass:

$$\chi, \psi, \omega, \chi_s, \psi_s, \omega_s, \chi_{ss}, \psi_{ss}, \omega_{ss},$$

als Funktionen der s, a, b betrachtet, von der Klasse C' sind in der Umgebung

$$h_1 \leq s \leq h_0, \quad |a - a_0| \leq k, \quad |b - b_0| \leq l,$$

wobei s_1, s_0 , diejenigen Werte von s bedeuten, welche den Punkten P_1, P_0 entsprechen, $h_1 < s_1, s_0 < h_0$ ist, und k, l positive Zahlen sind, die

von h_1, h_0 abhängen. Man sucht nun auf einer der Extremalen (15) $C_{a,b}$, welche der Extremale $C_{a_0, b_0} = P_1 P_0$ benachbart ist, einen Punkt $P(s)$ und eine Richtung $\bar{\theta}, \bar{\varphi}, \bar{\psi}$ zu bestimmen, derart, dass sie zusammen mit der Richtung $\bar{\theta}, \bar{\varphi}, \bar{\psi}$ der positiven Tangente der Extremale $C_{a,b}$ in Punkte P den Erdmann-Weierstrassschen Bedingungen (14) genügen.

Zur Bestimmung der unbekannten $s, \cos \bar{\theta}, \cos \bar{\varphi}, \cos \bar{\psi}$ haben wir die folgenden vier Gleichungen:

$$(16) \quad \begin{cases} \chi_1 = F_x(x, y, z; \theta, \varphi, \psi) - \bar{F}_x(x, y, z; \cos \bar{\theta}, \cos \bar{\varphi}, \cos \bar{\psi}) = 0, \\ \phi_1 = F_y - \bar{F}_y = 0, \\ z_1 = F_z - \bar{F}_z = 0, \\ \Phi = \cos^2 \bar{\theta} + \cos^2 \bar{\varphi} + \cos^2 \bar{\psi} - 1 = 0, \end{cases}$$

wobei die x, y, z durch $\chi(s, a, b), \phi(s, a, b), \omega(s, a, b)$ ersetzt sind und infolgedessen χ_1, ϕ_1, z_1 Funktionen von $s, a, b, \cos \theta, \cos \varphi, \cos \psi, \cos \bar{\theta}, \cos \bar{\varphi}, \cos \bar{\psi}$ sind. Zur Bildung der Funktionaldeterminante hat man

$$\begin{aligned} \frac{\partial}{\partial s} F_x &= F_x, & \frac{\partial}{\partial s} F_y &= F_y, & \frac{\partial}{\partial s} F_z &= F_z, \\ \frac{\partial \chi_1}{\partial s} &= F_x - \bar{F}_{x'x} \cos \theta - \bar{F}_{x'y} \cos \varphi - \bar{F}_{x'z} \cos \psi, \\ \frac{\partial \phi_1}{\partial s} &= F_y - \bar{F}_{y'x} \cos \theta - \bar{F}_{y'y} \cos \varphi - \bar{F}_{y'z} \cos \psi, \\ \frac{\partial z_1}{\partial s} &= F_z - \bar{F}_{z'x} \cos \theta - \bar{F}_{z'y} \cos \varphi - \bar{F}_{z'z} \cos \psi, \\ \frac{\partial \chi_1}{\partial (\cos \bar{\theta})} &= -\bar{F}_{x'x'}, & \frac{\partial \chi_1}{\partial (\cos \bar{\varphi})} &= -\bar{F}_{x'y'}, & \frac{\partial \chi_1}{\partial (\cos \bar{\psi})} &= -\bar{F}_{x'z'}, \\ \frac{\partial \phi_1}{\partial (\cos \bar{\theta})} &= -\bar{F}_{y'x'}, & \frac{\partial \phi_1}{\partial (\cos \bar{\varphi})} &= -\bar{F}_{y'y'}, & \frac{\partial \phi_1}{\partial (\cos \bar{\psi})} &= -\bar{F}_{y'z'}, \\ \frac{\partial z_1}{\partial (\cos \bar{\theta})} &= -\bar{F}_{z'x'}, & \frac{\partial z_1}{\partial (\cos \bar{\varphi})} &= -\bar{F}_{z'y'}, & \frac{\partial z_1}{\partial (\cos \bar{\psi})} &= -\bar{F}_{z'z'}, \end{aligned}$$

wobei $F_x, F_y, F_z, F_x, \dots$ Funktionen von $\chi, \phi, \omega, \chi_s, \phi_s, \omega_s$, und die $\bar{F}_x, \bar{F}_y, \bar{F}_z, \bar{F}_x, \dots$ von $\chi, \phi, \omega, \cos \bar{\theta}, \cos \bar{\varphi}, \cos \bar{\psi}$ sind. Die Funktionaldeterminante nimmt folgende Form an:

$$\begin{vmatrix} F_x - \bar{F}_{x'x} \cos \theta - \bar{F}_{x'y} \cos \varphi - \bar{F}_{x'z} \cos \psi, & -\bar{F}_{x'x'}, & -F_{x'y'}, & -\bar{F}_{x'z'} \\ F_y - \bar{F}_{y'x} \cos \theta - \bar{F}_{y'y} \cos \varphi - \bar{F}_{y'z} \cos \psi, & -\bar{F}_{y'x'}, & -F_{y'y'}, & -\bar{F}_{y'z'} \\ F_z - \bar{F}_{z'x} \cos \theta - \bar{F}_{z'y} \cos \varphi - \bar{F}_{z'z} \cos \psi, & -\bar{F}_{z'x'}, & -F_{z'y'}, & -\bar{F}_{z'z'} \\ 0, & 2 \cos \theta, & 2 \cos \bar{\varphi}, & 2 \cos \bar{\psi} \end{vmatrix}.$$

Multipliziert man die drei letzten Kolonnen bezüglichsweise mit $\cos \bar{\theta}$, $\cos \bar{\varphi}$, $\cos \bar{\psi}$ und addiert die so erhaltenen Produkte von der dritten und vierten zu den entsprechenden der zweite, so findet man mit Berücksichtigung der (6')

$$\frac{2}{\cos \bar{\theta}} \begin{vmatrix} F_x - \bar{F}_{x'x} \cos \theta - \bar{F}_{x'y} \cos \varphi - \bar{F}_{x'z} \cos \psi, & \bar{F}_{x'y'}, & \bar{F}_{x'z'} \\ F_y - \bar{F}_{y'x} \cos \theta - \bar{F}_{y'y} \cos \varphi - \bar{F}_{y'z} \cos \psi, & \bar{F}_{y'y'}, & \bar{F}_{y'z'} \\ F_z - \bar{F}_{z'x} \cos \theta - \bar{F}_{z'y} \cos \varphi - \bar{F}_{z'z} \cos \psi, & \bar{F}_{z'y'}, & \bar{F}_{z'z'} \end{vmatrix}$$

und nach Entwicklung derselben hat man wegen der (7) und (6)

$$\frac{2\bar{F}_1}{\cos \bar{\theta}} \left\{ (F_x - (\bar{F}_{x'x} \cos \theta + \bar{F}_{x'y} \cos \varphi + \bar{F}_{x'z} \cos \psi)) \cos^2 \bar{\theta} \right. \\ + (F_y - (\bar{F}_{y'x} \cos \theta + \bar{F}_{y'y} \cos \varphi + \bar{F}_{y'z} \cos \psi)) \cos \bar{\theta} \cos \bar{\varphi} \\ \left. + (F_z - (\bar{F}_{z'x} \cos \theta + \bar{F}_{z'y} \cos \varphi + \bar{F}_{z'z} \cos \psi)) \cos \bar{\theta} \cos \bar{\psi} \right\} = -2\bar{F}_1 \Omega,$$

wenn

$$\Omega = \cos \theta F_x(\bar{\theta}, \bar{\varphi}, \bar{\psi}) + \cos \varphi F_y(\bar{\theta}, \bar{\varphi}, \bar{\psi}) + \cos \psi F_z(\bar{\theta}, \bar{\varphi}, \bar{\psi}) \\ - \cos \bar{\theta} F_x(\theta, \varphi, \psi) - \cos \bar{\varphi} F_y(\theta, \varphi, \psi) - \cos \bar{\psi} F_z(\theta, \varphi, \psi)$$

gesetzt ist. Daraus hat man das Resultat:

Wenn die Bedingungen $\bar{F}_1 \neq 0$, $\Omega_0 \neq 0$ erfüllt sind, wobei Ω_0 den Wert von Ω im Punkte P_0 bedeutet, so lassen sich die Gleichungen (16) in der Umgebung des Punktes s_0 , a_0 , b_0 , $\cos \bar{\theta}_0$, $\cos \bar{\varphi}_0$, $\cos \bar{\psi}_0$ eindeutig nach s , $\cos \bar{\theta}$, $\cos \bar{\varphi}$, $\cos \bar{\psi}$ auflösen, und die Lösungen:

$$s = s(a, b), \quad \cos \bar{\theta} = \bar{\theta}(a, b), \quad \cos \bar{\varphi} = \bar{\varphi}(a, b), \quad \cos \bar{\psi} = \bar{\psi}(a, b)$$

sind von der Klasse C' in der Umgebung von $a = a_0$, $b = b_0$ und es werden die folgenden Bedingungen erfüllt:

$$\cos \bar{\theta}_0 = \bar{\theta}(a_0, b_0), \quad \cos \bar{\varphi}_0 = \bar{\varphi}(a_0, b_0), \quad \cos \bar{\psi}_0 = \bar{\psi}(a_0, b_0), \quad s_0 = s(a_0, b_0).$$

Nach der gemachten Voraussetzung ist, wenn $\bar{F}_1 > 0$ ist,

$$F_1(\chi(s(a, b), a, b), \quad \psi(s(a, b), a, b), \quad \omega(s(a, b), a, b); \\ \bar{\theta}(a, b), \quad \bar{\varphi}(a, b), \quad \bar{\psi}(a, b)) > 0$$

für alle Werte der a , b , welche die Ungleichungen:

$$|a - a_0| \leq k \quad |b - b_0| \leq l$$

erfüllen. Wenn man überdies voraussetzt, dass:

$$F_{x'x'}(\theta, \varphi, \psi), \quad F_{y'y'}(\theta, \varphi, \psi), \quad F_{z'z'}(\theta, \varphi, \psi) \geq 0, \\ F_{x'x'}(\bar{\theta}, \bar{\varphi}, \bar{\psi}), \quad F_{y'y'}(\bar{\theta}, \bar{\varphi}, \bar{\psi}), \quad F_{z'z'}(\bar{\theta}, \bar{\varphi}, \bar{\psi}) \geq 0,$$

so kann man das folgende System :

$$(17) \quad \begin{cases} \frac{d\bar{x}}{ds} = \cos \bar{\theta}, & \frac{d\bar{y}}{ds} = \cos \bar{\varphi}, & \frac{d\bar{z}}{ds} = \cos \bar{\psi}, \\ \frac{d\bar{\theta}}{ds} = -\frac{d^2\bar{x}}{ds^2} \cdot \frac{1}{\sqrt{1-\cos^2\bar{\theta}}}, & \frac{d\bar{\varphi}}{ds} = -\frac{d^2\bar{y}}{ds^2} \cdot \frac{1}{\sqrt{1-\cos^2\bar{\varphi}}} \\ \frac{d\bar{\psi}}{ds} = -\frac{d^2\bar{z}}{ds^2} \cdot \frac{1}{\sqrt{1-\cos^2\bar{\psi}}} \end{cases}$$

integrieren, wobei :

$$\frac{d^2\bar{x}}{ds^2}, \quad \frac{d^2\bar{y}}{ds^2}, \quad \frac{d^2\bar{z}}{ds^2}$$

mit Hülfe der Gleichungen (8) und der

$$\bar{x}'\bar{x}'' + \bar{y}'\bar{y}'' + \bar{z}'\bar{z}'' = 0$$

als Funktionen von $\bar{x}, \bar{y}, \bar{z}; \bar{x}', \bar{y}', \bar{z}'$ ausgedrückt sind, weil es mindestens eine der Determinanten :

$$\begin{vmatrix} \bar{F}_{x'x'}, & \bar{F}_{x'y'}, & \cos \bar{\theta} \\ \bar{F}_{y'x'}, & \bar{F}_{y'y'}, & \cos \bar{\varphi} \\ \bar{F}_{z'x'}, & \bar{F}_{z'y'}, & \cos \bar{\psi} \end{vmatrix}, \quad \begin{vmatrix} \bar{F}_{x'x'}, & \bar{F}_{x'z'}, & \cos \bar{\theta} \\ \bar{F}_{y'x'}, & \bar{F}_{y'z'}, & \cos \bar{\varphi} \\ \bar{F}_{z'x'}, & \bar{F}_{z'z'}, & \cos \bar{\psi} \end{vmatrix}, \quad \begin{vmatrix} \bar{F}_{x'y'}, & \bar{F}_{x'z'}, & \cos \bar{\theta} \\ \bar{F}_{y'y'}, & \bar{F}_{y'z'}, & \cos \bar{\varphi} \\ \bar{F}_{z'y'}, & \bar{F}_{z'z'}, & \cos \bar{\psi} \end{vmatrix}$$

verschieden von Null ist, und keiner der Nenner in den Gleichungen (17) gleich Null ist; denn wenn es z. B. $\cos \bar{\theta} = \pm 1$ wäre, so würde wegen (6') $\bar{F}_{x'x'} = 0$ sein, was gegen die gemachte Voraussetzung ist. Also man kann durch den Punkt $P(s)$ in der Richtung $\cos \bar{\theta}, \cos \bar{\varphi}, \cos \bar{\psi}$ eine Kurve konstruieren, welche durch die Gleichungen :

$$\bar{x} = \bar{\chi}(s, a, b), \quad \bar{y} = \bar{\varphi}(s, a, b), \quad \bar{z} = \bar{\omega}(s, a, b)$$

dargestellt wird. Der Parameter s wird so gewählt, dass auf dieser Kurve für den Wert $s=s(a, b)$ der Punkt P entspricht und infolgedessen wird sein :

$$\bar{\chi}(s, a, b) = \chi_1(a, b), \quad \bar{\varphi}(s, a, b) = \varphi_1(a, b), \quad \bar{\omega}(s, a, b) = \omega_1(a, b),$$

wobei s durch $s(a, b)$ ersetzt zu verstehen ist und

$$\chi_1(a, b) = \chi(s(a, b), a, b), \quad \varphi_1 = \varphi(s(a, b), a, b), \quad \omega_1 = \omega(s(a, b), a, b)$$

gesetzt ist. Wir haben also eine gebrochene Kurve $C_{a,b} + \bar{C}_{a,b}$, welche den Knickpunkt P enthält und auf welche der Parameter sich stetig ändert. Betrachtet man nun die a, b als Variablen, so erhält man eine Schar von gebrochenen extremalen Kurven des betrachteten Problems, welche die Lösung $P_1 P_0 P_2$ für $a=a_0, b=b_0$ enthalten.

Bemerkung über Veränderliche und Funktion,

von

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1. Veränderliche und Funktion sind beide fundamentalen Begriffe der modernen Analysis. In der gegenwärtigen Mathematik sind sie vom mengentheoretischen Standpunkte aus so streng definiert, dass keine Dunkelheit daran bleiben lassen zu sein scheint. Aber man kann durch mathematische Definition eines Begriffs nicht sogleich verstehen, wie er als Gegenstand der Wissenschaft möglich ist. Logische Begründung antwortet erst auf diese Frage. Ich möchte im folgenden eine davon für die Begriffe der Veränderlichen und der Funktion versuchen.

Historisch betrachtet, hat sich der Funktionsbegriff bei älteren Mathematikern, wie Leibniz und den Bernoullis, immer nur an einzelnen Beispielen, den Potenzen, trigonometrischen Funktionen u. dgl. gefunden. Allgemeinen Formulierungen begegnen wir zuerst im 18. Jahrhundert bei Euler. Seine Definition der Funktion lautet folgendermassen nach der Ausdrucksweise Dirichlets: Ist in einem Intervalle jedem einzelnen Werte x ein bestimmter Wert y zugeordnet, dann soll y eine Funktion von x heissen. Da ist ohne Zweifel immer eine stetige Funktion gedacht⁽¹⁾. Im 19. Jahrhundert hat sich einerseits die Theorie der Funktionen eines komplexen Argumentes in den Händen von Cauchy, Riemann und Weierstrass zu einem grossartigen Systeme entwickelt, während die Theorie der Funktionen eines reellen Argumentes andererseits einer strengen Durchsicht und feinen Bearbeitung sich unterworfen, und endlich durch den mengentheoretischen Gesichtspunkt eine schöne Grundlegung erhalten hat. Jetzt ist nicht nur einer stetigen, sondern einer unstetigen Funktion die Bedeutung anerkannt, und hat der Begriff der Funktion einen sehr allgemeinen Inhalt. Nach dieser heutigen Ansicht ist die Veränderliche ein Symbol für mehr als einen, im allgemeinen unendlich viele Zahlenwerte. Die Menge der Zahlen, deren Werte eine Veränderliche annehmen kann, heisst das Gebiet der Veränderlichen. Dies braucht nicht immer stetig zu sein, sondern es kann auch unstetig

(1) Klein, Elementarmathematik vom höheren Standpunkte aus I, S. 414.

sein. Wenn ein oder mehrere Zahlenwerte von einer Veränderlichen y für jeden Wert von einer andern Veränderlichen x im Gebiete der letztern bestimmt werden, und überdies die Beziehung dieser Zuordnung als eine Regel dargestellt wird, so heisst y Funktion von x . Die Funktion ist durch diese Regel definiert. Klein gibt den folgenden allgemeinen Ausdruck, damit er geometrische Gebilde auch umfassen könne: Man nennt y eine Funktion von x , wenn jedem Elemente einer Menge von Dingen (Zahlen oder Punkten) x ein Element einer Menge y zugeordnet ist⁽¹⁾.

Die heutige Definition der Funktion, welche ich oben erwähnt habe, ist ganz allgemein; danach ist die Funktion nichts weiter als „Correlation of series“, wie Russell deutet⁽²⁾. Wie der Begriff der Funktion solchermassen sich verallgemeinert hat, so hat sich auch der der Veränderlichen geklärt. Früher war die Veränderliche eine veränderliche Grösse, welche verschiedene Zahlenwerte annimmt. Aber welche Realität hat die veränderliche Grösse ausser den davon angenommenen Werten? In der gegenwärtigen Mathematik erkennt man Zahl, nicht Grösse, als den eigentlichen Gegenstand an. Die Veränderliche ist keine Grösse, sondern bloss ein Symbol für eine Menge von Zahlen. Auf diese Weise ist der Begriff der Veränderlichen formal geklärt. Aber dieser Formalismus macht uns um so stärker die Notwendigkeit der logischen Klarstellung des Begriffsinhalts empfinden. Das Symbol setzt immer etwas zu Symbolisierendes ausser ihm voraus. Der Begriff der Veränderlichen kann nicht blosses Symbol sein. Nur x , y und desgleichen sind Symbole. Die Veränderliche selbst muss das durch solches Symbolisierte als seinen Begriffsinhalt haben. Die als ihr Gebiet bestimmte Zahlenmenge macht nur den Umfang des Begriffs der Veränderlichen aus. Die Mathematik betrachtet eine Veränderliche hauptsächlich von der Seite des Umfangs allein; ihr eigentlicher Gegenstand ist zwar Zahl. Aber logisch muss eher der Begriffsinhalt zunächst bestimmt werden. Und mit dieser inhaltlichen Bestimmung des Begriffs der Veränderlichen wird man auch für denjenigen der Funktion eine inhaltsvollere Bestimmung erhalten als die oben erwähnte.

2. Es geschieht nur selten, dass eine Veränderliche eine endliche Menge der Zahlenwerte als ihr Gebiet hat. Im allgemeinen nimmt sie eine unendliche Menge derselben, wie es in der oben genannten Definition

(¹) Klein, Op. cit., S. 449.

(²) Russell, The Principles of Mathematics I, p. 263.

angedeutet ist. Sogar umfasst dieser letztere Fall den ersteren, weil eine endliche Menge sich als Beschränkung der unendlichen ergibt. Wir müssen also bloss den Begriff der Veränderlichen als des Symbols für eine unendliche Menge der Zahlenwerte betrachten. Nun ist es klar, dass eine unendliche Menge sich nicht auf distributive Weise definieren lässt. Man kann sie nicht durch die Aufzählung der Elemente bestimmen, sondern nur durch die Hinweisung auf den gemeinsamen Charakter derselben. Nämlich lässt sie sich bloss in kollektiver Weise definieren. Aber der gemeinsame Charakter der das Gebiet einer Veränderlichen ausmachenden Zahlen kann nichts anderes als dass sie alle diesem Gebiete angehören. Wenn etwa eine Veränderliche x die unendliche Menge aller reellen Zahlen grösser als a , und kleiner als b symbolisiert, so ist es der gemeinsame Charakter dieser Zahlenwerte, dass sie grösser als a und kleiner als b sind. Solcher gemeinsame Charakter ist logisch nichts als der Begriffsinhalt der betrachteten Zahlen. Das Symbol x steht für den Begriff der Zahl des Gebietes. Also ist eine Veränderliche der ihren Zahlenwerten gemeinsame Begriffsinhalt. Aber der allgemeine Begriff ist überhaupt nicht ein blosses Abstraktionsprodukt, wie die gemeine Logik lehrt, sondern vielmehr das dem Besonderen zu Grunde liegende Allgemeine. Für uns ist zwar das erstere vor, und das letztere nach; jedoch an sich ist dies vor, und jenes nach. Es besteht an sich, unabhängig von uns, oder nach Russells Terminologie subsistiert es, ob es vom Besonderen abstrahiert sein mag oder nicht. Das Besondere ist erst möglich durch eine Beschränkung des Allgemeinen. Die Veränderliche ist ein Allgemeines, durch dessen Spezifizierung einzelne Zahlenwerte sich ergeben. Der Mathematiker dürfte vielleicht vom formalistischen Standpunkte meinen, dass solches Allgemeine für die Definition der Veränderlichen unnötig sei, denn eine Veränderliche kann vollständig bestimmt werden, wenn es anders sich entscheiden lässt, ob eine Zahl ihrem Gebiete angehöre oder nicht. Aber bloss solche Entscheidung lässt nicht sogleich eine Veränderliche entstehen. Sie ist erst möglich, wenn das durch das Gebiet dargestellte Allgemeine als Denkobjekt gedacht ist. Diesen Umstand kann man deutlicher an der abhängigen Veränderlichen erkennen; denn sie ist nicht durch ihre einzelnen Zahlenwerte bestimmt, wie eine unabhängige Veränderliche, sondern nur mittels der Funktion. Diese ist eine allgemeine Regel, nach welcher Zahlenwerte der abhängigen Veränderlichen durch die der unabhängigen bestimmt werden. Die abhängige Veränderliche wird nicht anders als mittels dieser allgemeinen Regel bestimmt. Daher ist es kein Zweifel, dass sie ein allgemeiner

Begriff ist. Jetzt ist es allgemein festgestellt, dass eine Veränderliche das durch ihr Gebiet dargestellte Allgemeine ist, als dessen Besonderung einzelne Zahlenwerte sich ergeben. Das eben betrachtete Gebiet muss immer stetig sein; im Gegenteil ist es doch das Verdienst der gegenwärtigen Analysis, dass eine diskontinuierliche Veränderliche berücksichtigt ist. Aber das Unstetige ist logisch Beschränkung des Stetigen. Vom konkreten Gesichtspunkte kommt nur das letztere in Betracht. Und eine komplexe Veränderliche lässt sich ohne Weiteres als Zusammensetzung der reellen Veränderlichen betrachten. Also muss eine stetige, reelle Veränderliche als das Allgemeine im oben besprochenen Sinne vom transzendental-logischen Standpunkte betrachtet werden.

3. In den früheren Schriften habe ich eine logische Genesis des Zahlenbegriffs, von den natürlichen Zahlen bis zu den reellen Zahlen, versucht⁽¹⁾. Danach sind die Zahlen Produkte der Selbstentwicklung eines eigentümlichen Apriorischen. In ihrem Hintergrund ist es eine Erlebniseinheit, welche durch Zahlen objektiv konstruiert wird. Einzelne Zahlen sind Konstruktionselemente der sich selbst entwickelnden Ureinheit. Die ersteren sind möglich bloss auf dem Grunde der letzteren. Diese ist, wenn sie begrifflich objektiviert ist, nicht Zahlen, sondern die Zahl. Zahlen setzen die Zahl als ihren Grund voraus. Die Zahl ist das Substratum der Zahlen. Die als die Zahl objektivierte Ureinheit ist das Allgemeine, als dessen Beschränkung das Besondere sich ergibt. Also sind Zahlen dem Wesen nach nichts mehr als Zahlenwerte der Veränderlichen. Es ist eine abstrakte Ansicht, Zahlen bloss als solche zu betrachten. Konkreter müssen sie als Zahlenwerte der veränderlichen Zahl sein. Ihr Gebiet läuft stetig von $-\infty$ bis $+\infty$. Ein eingeschränktes Gebiet, etwa grösser als a und kleiner als b , oder auch ein unstetiges, ist nichts weiter als Beschränkung desselben. Die Zahl erzeugt Zahlen durch ihre Selbstentwicklung von $-\infty$ bis $+\infty$. Und umgekehrt kann man auch sekundär denken, dass die Zahl sich von $+\infty$ bis $-\infty$ entwickelt⁽²⁾. Ferner kann man sich denken irgend eine Zusammensetzung von beiden zunehmenden und abnehmenden Fortgängen. Daraus kann man ersehen, dass die Veränderliche nicht erst aus einzelnen Zahlen gebildet wird, sondern ihren Möglichkeitsgrund im Wesen der Zahl selbst besitzt. Im allgemeinen kann mathematisches Denken nur das *apriori* Grund gelegte

(1) Zur philosophischen Grundlegung der natürlichen Zahlen, Tôhoku Math. Journ. Vol. 7. Über die logische Grundlage des Zahlenkontinuums, T. M. J. Vol. 9.

(2) Vgl. Zur logischen Begründung der negativen und der imaginären Zahlen, T. M. J. Vol. 11.

als Gegenstand setzen. Der formalistische Nominalismus, welcher diesen Hauptpunkt übersieht, ist philosophisch unberechtigt. Eine Veränderliche begründet sich auf dem Wesen der Zahl. Ohne diese könnte jene nicht als mathematischer Gegenstand entstehen.

Die Veränderliche hat ihren Grund in der Ureinheit, welche der Erzeugung der Zahlen zu Grunde liegt. Sie ist ihre Objektivierung. Wenn dem so ist, kann auch die Funktion nicht bloss Regel der Zuordnung der Zahlenwerte zweier Veränderlichen zu einander sein. Sie muss auch den Rechtsgrund im Wesen des mathematischen Denkens selbst haben. Nun ist Denken im Allgemeinen Begründung. Nach dem Satze des Grundes fordert jedes Denkobjekt seine Begründung durch anderes Objekt. Eine Veränderliche auch lässt sich konkret als Denkobjekt nicht ohne Begründung denken. Sie muss ihrem Wesen nach durch eine andere Veränderliche begründet werden. Die Funktion ist es, die diese Forderung erfüllt. Sie ist eine Regel, nach welcher die Selbstentwicklung einer Veränderlichen durch ihre Abhängigkeit von der anderen bestimmt wird. Diese letztere, die in der Mathematik als unabhängige Veränderliche bezeichnet ist, muss als diejenige betrachtet sein, welche nicht mehr durch eine dritte Veränderliche begründet zu werden braucht. Daher kommt es, dass die unabhängige Veränderliche immer als ein in einem gewissen Gebiete eingeschränkter Teil der Zahl selbst angenommen ist. Jede Veränderliche wird in ihrer Selbstentwicklung durch ihre Abhängigkeitsrelation zu derjenigen der Zahl selbst begründet. Diese Abhängigkeitsrelation ist Funktion. Dies ist nicht bloss eine äusserliche Beziehung zwischen zwei Reihenfolgen der Zahlenwerte wie die formalistische Mathematik annimmt. Solche Ansicht ist logisch unberechtigt, wie sie auch in der Mathematik zu empfehlen sein mag. Funktion ist das konkreteste Produkt des die Zahlen erzeugenden mathematischen Denkens. Veränderliche und Funktion sind Krone des Zahlbegriffs.

4. Die Bestimmung der Funktion als Grundes der Zahlenerzeugung wird durch den Differentialquotient erfüllt. Ich habe an andern Orte⁽¹⁾ den logischen Grund des Differentials erläutert. In der heutigen Mathematik ist das Differential als eine solche bestimmte unendlich kleine Grösse wie Leibniz gemeint hat, verworfen, und an dessen Stelle der Differentialquotient als Grenze des Quotienten zweier verschwindenden Zahlen als Grundbegriff der ganzen Analysis gestellt. Das Differential ist nichts

(¹) T. M. J. Vol. 9.

weiter als eine solche verschwindende Zahl. Es ist etwas, das nicht ist, sondern wird. Aber solches den Differentialquotienten tragende Zahlelement ist, vom logischen Standpunkte aus gesehen, das sogenannte Erzeugungselement. Der Differentialquotient stellt das Gesetz der Erzeugung dar. Und dies Gesetz wird durch die Funktion bestimmt, nach welcher eine Veränderliche durch eine andere begründet wird. Es ist in der Tat falsch, dass jede stetige Funktion einen Differentialquotienten überall hat, vielmehr ist es der Verdienst der neueren Mathematik, dass dieser einst als selbstverständlich anerkannte Satz durch ein etwa von Weierstrass gegebenes Beispiel verneint wird, und dass Stetigkeit der Funktion und ihre Differenzierbarkeit unterschieden werden. Beide sind nicht dasselbe, sondern die erstere hat einen weiteren Begriffsumfang als die letztere. Aber die mathematische Undifferenzierbarkeit scheint nicht zu bedeuten, dass eine Funktion keinen Differentialquotienten hat, sondern vielmehr dass, der Wert des Differentialquotienten sich am jeden Orte unstetig verändert, oder oszilliert. Soweit eine Veränderliche durch eine Funktion begründet ist, muss das Erzeugungsgesetz logisch notwendig durch einen Differentialquotienten dargestellt werden. Ob dieser gleich mathematisch nicht festzustellen ist, so existiert er doch gewiss. Er ist der wahre Erzeuger der endlichen Grösse, wie Natorp bemerkt⁽¹⁾.

Aus dem oben Betrachteten erhellt sich auch der Sinn des Integrals. Dies hat in der heutigen Mathematik eine selbständige Bedeutung, unabhängig davon, dass es als Gegensatz des Differentials zu betrachten ist. Es ist die Grenze einer Summe der unendlich vielen Produkte, Faktor von deren jedem unendlich klein wird. Diese Definition scheint mathematisch vollkommen klar. Aber die Summationsgrenze der unendlich vielen Glieder schliesst überhaupt eine unerfüllbare Forderung des Denkens. Ihre Termini sind nicht distributiv zu ernennen. Man kann sie bloss kollektiv durch ihren allgemeinen Charakter definieren. Ferner ist auch ihre Summe nicht extensiv, sondern bloss intensiv zu bestimmen. Sie kann nicht anders als durch die Selbstentwicklung der Erzeugungseinheit erreicht werden. $f'(x) = \frac{dy}{dx}$ ist das Gesetz dieser Selbstentwicklung, und $f'(x)dx$ bedeutet logisch das Differential dy , welches dieses Gesetz trägt. Das von ihm erzeugte intensionale Ganze ist das Integral $\int f'(x)dx = f(x) = y$. Cohens Ansicht, dass das Integral ein Erzeugungswert des Endlichen ist, während das Differential Erzeu-

(1) Natorp, Die logischen Grundlagen der exakten Wissenschaften, S. 215.

gungseinheit anstatt der extensiven Masseinheit darstellt⁽¹⁾, ist logisch zu rechtfertigen, ob sie zwar mathematisch ungereimt scheinen mag. Russell, der sie streng getadelt hat⁽²⁾, konnte nicht ihre Tiefe würdigen.

5. Das oben betrachtete Wesen der Zahl als Veränderlicher lässt uns auch sogleich die Beziehung zwischen Zahl und Grösse einsehen. Zahlen sind, wie oben bemerkt, Spezifikationsprodukte der allgemeinen Zahl. Die letztere ist der ursprüngliche Grund der ersteren, durch welche erst sie möglich sind. Zahlen aber, die auf dem Grunde der allgemeinen Zahl konkret gedacht sind, stellen nicht bloss wie viele von etwas anderem dar, sondern wie gross von der Zahl selbst. Die Zahl ist Substratum von Zahlen⁽³⁾. Die so gedeutete Zahl ist nichts anderes als eine Grösse. In der älteren Mathematik sind Zahl und Grösse nicht deutlich unterschieden, sondern mit einander verwechselt worden. Die Veränderliche auch ist als veränderliche Grösse gedacht worden. Dagegen sind beide in der gegenwärtigen Mathematik ganz deutlich geschieden. Heute soll die reine Mathematik Lehre von Zahlen, nicht von Grössen sein. Die letzteren sind Gegenstand von der angewandten Mathematik. Nun ist das Folgende als Eigenschaften der Grössen erkannt. Erstens muss eine Grösse gleich wie, grösser oder kleiner als eine andere sein. Dies Verhältnis macht die erste notwendige Eigenschaft der Grösse. Zweitens müssen mehrere Grössen mit einander additiv sein. Zwei oder mehr gleichnamige Grössen zusammengenommen müssen eine bestimmte Grösse als ihre Summe geben. Drittens muss eine Grösse stetig veränderlich sein, so dass das sogenannte Archimedische Prinzip davon gilt. Nun sind empirische Grössen die eigentlich räumliche Grössen sein müssen, bleiben immer annähernd; sie können nicht mathematische Genauigkeit erreichen. Aber auch angewandte Mathematik als Lehre von Grössen verlangt mathematische Genauigkeit, denn sie muss alle Beziehungen der reellen Zahlen, von denen die reine Mathematik handelt, auf die der Grössen anwendbar machen. Wie ist es aber möglich, dass alle Beziehungen der reellen Zahlen zugleich als die der Grössen gedeutet werden? Der Grund davon ist, dass die Zahl selbst als allgemeines Substratum der Zahlen die Eigenschaften der Grösse besitzt, wie sich aus dem Obigen erhellen mag. Im Laufe der Zahlenerzeugung steht die nachkommende Zahl im Verhältnis von Grössersein zu jeder vorhergehenden, die letztere in dem von Kleinersein zu der ersteren, und ist jede Zahl sich selbst

(1) Cohen, Das Prinzip der Infinitesimalmethode und seine Geschichte, S. 146.

(2) Russell, Op. cit. I, p. 338—345.

(3) Vgl. Natorp, Op. cit. S. 200 ff.

gleich, wievielmals sie auch gedacht werden mag. Ferner folgt ohne Weiteres aus dem Wesen der Zahlen ihre Additivität und Stetigkeit. Die Zahl besitzt gewiss die oben genannten Haupteigenschaften der Grösse. Zahlen drücken also nicht bloss wie viele von etwas anderem, sondern wie gross von der Zahl selbst aus. Die Anwendbarkeit der Zahlen auf Grössen ist durch dieses Wesen der Zahl gesichert. Vom logischen Standpunkte muss man sogar sagen, dass die Grösse konkretes Substratum der Zahlen ist. Andere Arten der Grössen, oder Grössen im eigentlichen Sinne, lassen sich soweit mittels Zahlen bezeichnen, als sie mit der Zahl als Grösse übereinstimmen.

Solche Eigenschaften der Grösse, wie oben betrachtete, gehören ohne Zweifel nur der Zahl als stetiger Veränderlicher an. Das System der natürlichen, oder der rationalen Zahlen kann nicht als Grösse gedeutet werden. Erst das Kontinuum der reellen Zahlen kann als solches betrachtet werden, indem es das Substratum der einzelnen Zahlen darstellt. Dies lässt uns auf der andern Seite verstehen, warum die Stetigkeit einst als Eigenschaft der Grössen, nicht der Zahlen gedacht wurde, und weshalb irrationale Zahlen bloss als für die Grössenmessung auf der Grundlage der Anschauung annehmbar erkannt wurde⁽¹⁾. Obgleich diese Ansicht von der gegenwärtigen Mathematik verworfen ist, muss man ihr gewisse Berechtigung zugeben, wie Natorp bemerkt⁽²⁾, wenn man den oben angedeuteten Punkt erwägt. In Wahrheit ist die Zahl selbst eine Grösse, oder man kann sagen, der reine Begriff der Grösse. Das stetige System der Zahlen als konkretes Produkt der Zahlenerzeugung hat den Möglichkeitsgrund der Grössenmessung in sich. Wie Natorp sagt, muss das Fundament der zählbaren Grösse der reine Grundbegriff der Grösse sein⁽³⁾.

(1) Hankel, Theorie der komplexen Zahlensysteme, S. 46—47.

(2) Natorp, Op. cit. S. 204.

(3) Natorp, Op. cit. S. 204.

Theory of Continuous Set of Points

(finished : for the first portion see Vol. 12, pp. 43—158).

by

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CHAPTER II.

CONTINUOUS SET WITH PRINCIPAL POINTS OF THE N^{TH} ORDER.

Part I.

Continuous Set with Biprincipal Points.

In the previous chapter, we have investigated the properties, constitution, and classification of the continuous sets with principal points, namely of those of the second and third kinds. Here we shall treat the continuous sets having no pair of principal points, namely those of the first kind.

Definition 13. When a continuous set has two distinct points A, B , such that there are two and only two components having (A, B) as a pair of principal points and the sum of these components is the set itself, the points A, B are called a pair of biprincipal points of the set.

There are many continuous sets having a pair of biprincipal points. For example, 1), in any closed Jordan curve, any two points of it form always a pair of biprincipal points of the set; and 2), in the continuous set defined by the equations

$$(i) \quad y = \sin \frac{\pi}{1-x}, \quad z=0 \quad (0 \leq x < 1)$$

$$(ii) \quad y = (+1, -1), \quad z=0 \quad (x=1)$$

$$(iii) \quad x^2 - x + z^2 = 0, \quad y=0 \quad (0 \leq z \leq \frac{1}{2}),$$

the point $A(0, 0, 0)$ and any point on the line (ii) form always a pair of biprincipal points of the set.

Henceforth instead of "a pair of principal points" we shall often use the words "a pair of uniprincipal points," in contrast with "a pair of biprincipal points."

A continuous set with biprincipal points A, B is denoted by $M^{(2)}(A, B)$, and its constituent components by $M_1(A, B)$ and $M_2(A, B)$.

We shall first examine whether a continuous set with a pair of biprincipal points may have a pair of principal points. In the case where the constituent components have no common continuous component containing one of its biprincipal points, the condition for that is stated as follows.

Theorem 140. The necessary and sufficient condition that a continuous set with a pair of biprincipal points should have a pair of principal points is that each of its constituent components should have a compound principal point having all of their common points as its conjugate principal ones.

Thus the two constituent components of the set having the said property must be both singular.

Proof. I. The condition is necessary.

Suppose that the set has a pair of principal points (C, D) , then it is clear that the points C, D cannot belong to only one of the constituent components. Now assume that C belongs to $M_1(A, B)$ and D to $M_2(A, B)$, and take any common point Q . Since the sum of $M_1(C, Q)$ and $M_2(D, Q)$ must be the set itself, $M_1(C, Q)$ must contain all points of $M_1(A, B)$, other than the common points of the constituent components. But as $M_1(A, B)$ and $M_2(A, B)$ contain neither common continuous part containing A nor that containing B , there are an infinite number of non-common points in the neighborhood of A and B ; and $M_1(C, Q)$ is a continuous set containing all such points. Therefore it must contain A, B as their limiting points. Thus $M_1(C, Q)$ is identical with $M_1(A, B)$.

Similarly $M_2(D, Q)$ is identical with $M_2(A, B)$. Accordingly any common point Q is a conjugate principal point of C and D with respect to $M_1(A, B)$ and $M_2(A, B)$ respectively.

II. The condition is sufficient.

Denote by C, D the compound principal points of the two constituent components having the property stated in the theorem. Then any continuous component containing C, D must contain a common point Q and accordingly must contain the two components $M_1(C, Q)$ and $M_2(D, Q)$. But by hypothesis

$$M_1(C, Q) \equiv M_1(A, B),$$

$$M_2(D, Q) \equiv M_2(A, B).$$

Thus any continuous component containing C, D is the set itself. So (C, D) is a pair of principal points of the set.

From what has just been proved, it follows that $M_1(A, B)$ and $M_2(A, B)$ have singular systems of three points (A, B, C) and (A, B, D) respectively, A, B being the common points of the two components. Therefore they are both singular.

The previous condition was given for the particular set whose constituent components have no common continuous component containing biprincipal points. But to find the necessary and sufficient condition in general is more difficult, and we have first to investigate the constitution of the set in question.

Theorem 141. *If a continuous set with biprincipal points A, B has a pair of uniprincipal points (P, Q) , then each of its constituent components contains a singular components, and all other points are common to the both components.*

Proof. Since the sum of $M_1(A, P)$ and $M_2(A, Q)$ must be identical with the whole set, so at least $M_1(A, B) - M_1(A, P)$ is contained in $M_2(A, B)$. Similarly $M_1(A, B) - M_1(B, P)$ must also be contained in $M_2(A, B)$. Therefore all points of $M_1(A, B)$ other than the common points $\{P\} \equiv \emptyset$ of the both components $M_1(A, P)$ and $M_1(B, P)$ are contained in $M_2(A, B)$. Here we shall study the properties of this set of the common points Φ .

1. The set Φ is closed, since it is the set of common points of two continuous sets.

2. The set Φ may be decomposed into the three parts Ψ_1, Ψ_2 , and Ψ_3 , such that

(a) all points of Ψ_1 are non-conjugate principal points of A with respect to $M_1(A, P)$ and conjugate principal points of B with respect to $M_2(B, P)$;

(b) all points of Ψ_2 are conjugate principal points of A and B at the same time;

(c) all points of Ψ_3 are non-conjugate principal points of B while all of them are conjugate principal points of A .

The proof may be given as follows.

To begin with, some points of the set Φ cannot be the conjugate principal points of A with respect to $M_1(A, P)$, since, if all points of Φ were so, then, by Theorem 40a, they and accordingly the whole set $M_1(A, B)$ would be contained in $M_2(A, B)$, which is of course impossible. Therefore Φ consists of the two parts Ψ_1, Ψ_2' , the former of which consists of non-conjugate principal points of A , and the latter of conjugate principal ones of A with respect to $M_1(A, P)$. Similarly Φ may be de-

composed into two parts $\mathcal{Q}_1, \mathcal{Q}_2'$, the former of which consists of non-conjugate principal points of B , and the latter of conjugate principal ones of B with respect to $M_2(B, P)$. But since any point of Φ must be the conjugate principal point of at least one of A, B with respect to $M_1(A, P)$ and $M_2(B, P)$, all points of \mathcal{T}_1 are conjugate principal points of B and so is contained in \mathcal{Q}_2' ; and all points of \mathcal{Q}_1 are conjugate principal points of A and so is contained in \mathcal{T}_2' . Therefore Φ is decomposed into the three parts

$$\mathcal{T}_1, \mathcal{T}_2' - \mathcal{Q}_1 \equiv \mathcal{Q}_2' - \mathcal{T}_1 \equiv \mathcal{T}_2, \quad \mathcal{Q}_1 \equiv \mathcal{T}_3$$

having the said property.

3. The set $\mathcal{T}_2 + \mathcal{T}_3$ contains *all* the conjugate principal points of A with respect to $M_1(A, P)$ and the set $\mathcal{T}_1 + \mathcal{T}_2$ contains *all* the conjugate principal points of B with respect to $M_2(B, P)$.

Take any point R of \mathcal{T}_1 , then $M_1(A, R)$ contains no conjugate principal point of A since R is a non-conjugate principal one of A ; and so all conjugate principal points of A are contained in $M_1(B, R) \equiv M_1(B, P)$, and accordingly in Φ , which is the common part of $M_1(A, P)$ and $M_1(B, P)$. Thus $\mathcal{T}_2 + \mathcal{T}_3$ consists of all these principal points. Similarly for $\mathcal{T}_1 + \mathcal{T}_2$.

4. The set Φ is connected. For, by Theorems 36, 44, and 49, the aggregate of conjugate principal points of a certain principal point in any continuous set is always connected, so each of the sets $\mathcal{T}_1 + \mathcal{T}_2$ and $\mathcal{T}_2 + \mathcal{T}_3$ is connected, and accordingly their sum is also connected.

From 1 and 4, we may conclude that the set Φ is continuous.

5. The set Φ is singular. For, take any two points R and S from \mathcal{T}_1 and \mathcal{T}_3 respectively, and consider a set $M(R, S)$ contained in Φ . Then since R is a non-conjugate principal point and S a conjugate principal point of A with respect to $M_1(A, P)$, so $M(R, S)$ must contain all the conjugate principal points of A , that is, $\mathcal{T}_2 + \mathcal{T}_3$. But, on the other hand, since R is a conjugate principal point and S a non-conjugate principal one of B with respect to $M_2(B, P)$, so $M(R, S)$ must contain all the conjugate principal points of B , that is, $\mathcal{T}_1 + \mathcal{T}_2$. Therefore $M(R, S)$ is identical with Φ . Next take two points R and P from \mathcal{T}_1 and \mathcal{T}_2 respectively, and consider a set $M(P, R)$ contained in Φ , then by the same reasoning as before, $M(P, R)$ contains \mathcal{T}_2 and \mathcal{T}_3 and accordingly the point S . Therefore

(¹) That the component \mathcal{T}_2 exists is clear from the fact that it contains at least one point P .

$$M(P, R) \equiv M(R, S) \equiv \Phi.$$

Further if we take P and S from \mathcal{P}_2 and \mathcal{P}_3 , and consider a set $M(P, S)$, then as before we have

$$M(P, S) \equiv M(R, S) \equiv \Phi.$$

Therefore

$$\Phi \equiv M(R, S) \equiv M(P, S) \equiv M(P, R),$$

which shows that the set Φ is a singular one having a singular system of three points (R, P, S) .

Hence it follows that $M_1(A, B)$ contains a singular component $M_{s,1}(R, P, S)$. Similarly it may be proved that $M_2(A, B)$ contains also another singular component $M_{s,2}(R', Q, S')$. And from what has been stated in the beginning of this proof, all those points of the two constituent components which are not contained in the above singular components $M_{s,1}(R, P, S)$ and $M_{s,2}(R', Q, S')$ are common to the both components.

Cor. 1. If some of the common points of $M(A, C)$ and $M(B, C)$, components of a continuous set $M(A, B)$, are not the limiting ones of non-common part of them, then the common points form a singular set.

Cor. 2. In an ordinary set $M(A, B)$, all the common points of its components $M(A, C)$ and $M(B, C)$ are limiting ones of non-common points of them.

Theorem 142. In the above, the sum of the two singular components $M_{s,1}(R, P, S)$ and $M_{s,2}(R', Q, S')$ is identical with the given set $M_1(A, B) + M_2(A, B)$, even when $M_{s,1}(R, P, S)$ is not identical with $M_1(A, B)$, and $M_{s,2}(R', Q, S')$ with $M_2(A, B)$.

Proof. By Theorems 51 and 141, the two sets $M_1(A, B)$ and $M_2(A, B)$ may be decomposed into the three parts

$$\text{I } M_1(A, R) \quad M_{s,1}(R, P, S) \quad M_1(S, B)$$

$$\text{II } M_2(A, R') \quad M_{s,2}(R', Q, S') \quad M_2(S', B),$$

such that all common points of the singular part and the other parts are the principal points of the latter parts. Now any point T , which belongs to $M_1(A, R)$, but not to $M_{s,1}(R, P, S)$, belongs to $M_2(A, R') + M_{s,2}(R', Q, S')$ or $M_2(S', B)$, since, by the previous theorem, any point except those of the singular part must be common to the both components. But T cannot belong to $M_2(S', B)$, since, if so, $M_1(A, R) + M_2(S', B)$ would contain a component $M(A, B)$ containing neither $M_{s,1}(R, P, S)$ nor $M_{s,2}(R', Q, S')$, contrary to the hypothesis. Therefore T and accord-

ingly all points of $M_1(A, R)$ must belong to $M_2(A, R') + M_{s, 2}(R', Q, S')$. Now assume that $M_{s, 1}$ and $M_{s, 2}$ had no common point, then R and accordingly $M_1(A, R)$ would be contained in $M_2(A, R')$.⁽¹⁾ But, on the other hand, under the same assumption, by the same reasoning as above, we can prove that $M_2(A, R')$ would be contained in $M_1(A, R)$. Hence it follows that $M_1(A, R)$ and $M_2(A, R')$ are identical, but this is impossible unless the two singular components have certain common points.

Thus $M_{s, 1}$ and $M_{s, 2}$ have at least a common point T , and so the sum of them contains a component $M(P, Q)$, whence follows that

$$M_1(A, B) + M_2(A, B) \equiv M_{s, 1}(R, P, S) + M_{s, 2}(R', Q, S'),$$

since (P, Q) is a pair of principal points of the set $M_1(A, B) + M_2(A, B)$.

Theorem 143. *The necessary and sufficient condition that a continuous set with biprincipal points should have a pair of uniprincipal points is that each of the singular components contained in its constituent components should have at least one singular aggregate which has no common point with the other singular component.*

Proof. I. The condition is necessary.

Suppose that all singular aggregates of $M_{s, 1}(R, P, S)$ had common points with $M_{s, 2}(R', Q, S')$, then the aggregate $\mathfrak{A}_{1, P}$ containing P would have a common point R'' with $M_{s, 2}(R', Q, S')$. The sum of $M_{s, 1}(P, R'')$ and $M_{s, 2}(Q, R'')$ then must be identical with $M_{s, 1}(R, P, S) + M_{s, 2}(R', Q, S')$ since it contains a component $M(P, Q)$. But $M_{s, 1}(P, R'')$, being a component of $\mathfrak{A}_{1, P}$, contains no point of $M_{s, 1}(R, P, S) - \mathfrak{A}_{1, P}$; and moreover another component $M_{s, 2}(Q, R'')$ cannot wholly contain $M_{s, 1}(R, P, S) - \mathfrak{A}_{1, P}$, since, if so, it would also contain $\mathfrak{A}_{1, P}$ itself and so $M_{s, 1}(R, P, S)$ would be identical with $M_{s, 2}(R', Q, S')$. Thus the sum of $M_{s, 1}(P, R'')$ and $M_{s, 2}(Q, R'')$ does not contain certain points of $M_{s, 1}(R, P, S) - \mathfrak{A}_{1, P}$, which

(¹) Since $M_2(A, R')$ contains R , so it contains a component $M_2(A, R)$. If $M_1(A, R)$ were not contained in $M_2(A, R)$, then $M_1(A, R)$ would contain a point G of $M_{s, 2}(R', Q, S')$, not contained in $M_2(A, R)$, and thus

$$M_1(A, R) \equiv M_1(A, G) + M_1(G, R).$$

But

$$M_1(A, G) + M_{s, 2}(R', Q, S') \equiv M_2(A, R') + M_{s, 2}(R', Q, S'),$$

therefore

$$M_1(A, G) - M_{s, 2}(R', Q, S') \equiv M_2(A, R') - M_{s, 2}(R', Q, S').$$

So $M_1(A, G)$ must contain $M_2(A, R')$, since the common points of $M_2(A, R')$ and $M_{s, 2}(R', Q, S')$ are the limiting ones of $M_2(A, R') - M_{s, 2}(R', Q, S') \equiv M_1(A, G) - M_{s, 2}(R', Q, S')$; that is, $M_1(A, R)$ must contain $M_2(A, R')$ and so its component $M_2(A, R)$, but this is clearly absurd. Hence $M_2(A, R')$ contains $M_1(A, R)$.

contradicts the above result. Therefore in order that the set has a pair of principal points (P, Q) , it is necessary that at least one aggregate of $M_{S,1}(R, P, S)$ has no common point with $M_{S,2}(R', Q, S')$. Similarly for $M_{S,2}(R', Q, S')$.

II. The condition is sufficient.

Denote by $\mathfrak{A}_{1,r}$ the singular aggregate of $M_{S,1}(R, P, S)$ which has no common point with $M_{S,2}(R', Q, S')$; and by $\mathfrak{Q}_{2,s}$ another singular aggregate of $M_{S,2}(R', Q, S')$ having the same property. From $\mathfrak{A}_{1,r}$ and $\mathfrak{Q}_{2,s}$, take any points P_r and Q_s respectively and consider a component $M(P_r, Q_s)$. Since P_r belongs to $M_{S,1}(R, P, S)$ and Q_s to $M_{S,2}(R', Q, S')$, $M(P_r, Q_s)$ must contain a common point R'' of $M_{S,1}(R, P, S)$ and $M_{S,2}(R', Q, S')$. Therefore $M(P_r, Q_s)$ contains $M(P_r, R'')$ and $M(Q_s, R'')$. But P_r, R'' belonging to different aggregates of $M_{S,1}(R, P, S)$, $M(P_r, R'')$ must be identical with $M_{S,1}(R, P, S)$ itself. Similarly $M(Q_s, R'')$ must be identical with $M_{S,2}(R', Q, S')$. Thus we have

$$\begin{aligned} M(P_r, Q_s) &\equiv M(P_r, R'') + M(Q_s, R'') \\ &\equiv M_{S,1}(R, P, S) + M_{S,2}(R', Q, S') \\ &\equiv M_1(A, B) + M_2(A, B). \end{aligned}$$

That is, (P_r, Q_s) is a pair of uniprincipal points of the given set $M_1(A, B) + M_2(A, B)$.

Cor. 1. If a continuous set with biprincipal points has a pair of uniprincipal points, then it has an infinite number of pairs of them.

Cor. 2. The above set $M(P_r, Q_s) \equiv M_1(A, B) + M_2(A, B)$ cannot be singular.

For, take any point S'' in the set, if S'' belong to the component $M_{S,1}(R, P, S)$, then S'', P_r cannot be a pair of principal points; and if it belong to $M_{S,2}(R', Q, S')$, then S'', Q_s cannot be a pair of principal points. So the set can have no singular system of three points (Theorem 10). Therefore the set must be semi-singular.

From the previous theorem we have an important theorem.

Theorem 144. An ordinary continuous set with biprincipal points cannot have a pair of uniprincipal points.

Theorem 145. The necessary and sufficient condition that an aggregate of non-principal points of a continuous set $M(A, B)$ should be semi-continuous is that the set should have no pair of biprincipal points.

Proof. I. The condition is necessary.

Denote by Φ_A the aggregate of principal points containing A , and by Φ_B that containing B , and by Ψ that of non-principal points. If the

set $M(A, B)$ has a pair of biprincipal points (P, Q) , then since the set $M_1(P, Q) + M_2(P, Q)$ has a pair of principal points (A, B) , by Theorem 143, the set consists of two singular components $M_{s,1}(P', A, Q')$ and $M_{s,2}(P', B, Q')$, where P', Q' are non-principal points of the set $M(A, B)$.⁽¹⁾ Moreover the set $M(A, B)$ cannot contain a third component $M_3(P', Q')$ containing neither A nor B , since, if so, the set would contain a third component $M_3(P, Q)$.⁽²⁾ Therefore the two non-principal points P', Q' cannot be continuously connected in the aggregate \mathcal{P} , and so the aggregate \mathcal{P} is not semi-continuous. Thus in order that the aggregate \mathcal{P} may be semi-continuous it is necessary that the set has no pair of biprincipal points.

II. The conditions is sufficient.

Suppose that the set has no pair of biprincipal points, and take any two points P, Q from \mathcal{P} . The set $\phi_A + \mathcal{P}$ has a component having P, Q as a pair of principal points, denote it by $M_1(P, Q)$; and since $\phi_A + \mathcal{P}$ contains no conjugate principal point of A , so also $M_1(P, Q)$ does not. Similarly a component $M_2(P, Q)$, contained in $\mathcal{P} + \phi_B$, contains no conjugate principal point of B . Therefore if $M_1(P, Q)$ be identical with $M(P, Q)$, then it contains no principal point and so consists of the points of \mathcal{P} only, and accordingly P, Q are continuously connected in \mathcal{P} .

If $M_1(P, Q)$ be different from $M_2(P, Q)$ and yet at least one of them be free from the principal points of the set, then P, Q are also continuously connected in \mathcal{P} .

Lastly if $M_1(P, Q)$ be different from $M_2(P, Q)$ and both of them contain the principal points of the set, that is, if $M_1(P, Q)$ contain a principal point A' , and $M_2(P, Q)$ its conjugate principal points B' , then $M_1(P, Q) + M_2(P, Q)$ contains $M(A, B)$ itself, and so, by the same reasoning as in Theorem 141, we know that the set consists of

$$M_1(P, Q) \equiv M_1(P, R) + M_{s,1}(R, A, S) + M_1(S, Q),$$

and

$$M_2(P, Q) \equiv M_2(P, R') + M_{s,2}(R', B, S') + M_2(S', Q).$$

But since P, Q cannot be a pair of biprincipal points by hypothesis, so the set must contain other components having P, Q as a pair of principal

(1) For, if P' were a principal point of the set, then it would be a conjugate principal point of A or B (Theorem 3), and therefore one of $M_{s,1}(P', A, Q')$ and $M_{s,2}(P', B, Q')$ would be identical with the set $M(A, B)$, which is impossible. Similarly for the point Q' .

(2) $M_1(P, P')$ and $M_1(Q, Q')$ contain neither A nor B (Theorem 143) and therefore $M_1(P, P') + M_3(P', Q') + M_1(Q, Q') \supseteq M_3(P, Q)$ contains neither A nor B .

points, and among them there must be at least one which contains no principal point. For, if we suppose that there were no such component, then we would be led to a contradiction as will be seen from the following consideration.

To begin with, when any component $M(P, Q)$ contains A (or any conjugate principal point of B) it must contain a singular component $M_{s,1}(R, A, S)$,⁽¹⁾ and when it contains B (or any conjugate principal point of A), it must contain a singular component $M_{s,2}(R', B, S')$. Moreover if all components having P, Q as a pair of principal points contain a principal points of the set $M(A, B)$, by the same reasoning as in Theorem 142, we can prove that $M_{s,1}(R, A, S)$ and $M_{s,1}(R', B, S')$ have common points P', Q' , such that

$$M_{s,1}(R, A, S) \equiv M_{s,1}(P', A, Q'),$$

$$M_{s,1}(R', B, S') \equiv M_{s,2}(P', B, Q'),$$

and
$$M_{s,1}(P', A, Q') + M_{s,2}(P', B, Q') \equiv M(A, B).$$

Thus there are two components having P', Q' as a pair of principal points, whose sum is identical with the whole set. Now, in this case, we may prove that there is no third component having P', Q' as a pair of principal points. For, (i) if there were a third component $M_3(P', Q')$ having no principal point, then there would be a third component $M_3(P, Q)$ having no principal point in the set $M_1(P, P') + M_3(P', Q') + M_1(Q, Q')$, since $M_1(P, P')$ and $M_1(Q, Q')$ have no principal point; which contradicts the supposition. (ii). If there be a third component $M_3(P', Q')$ having a principal point (say, A), then it must be identical with $M_{s,1}(P', A, Q')$, by the same reasoning as in the foot note of this page.

Thus from the assumption that any component having P, Q as a pair of principal points always contains a principal points of the set, it follows that P', Q' would be a pair of biprincipal points, which contradicts our hypothesis that the set has no pair of biprincipal points. So among the set $\{M(P, Q)\}$ there must be at least one containing no principal point.

(1) When $M_3(P, Q)$ contains A , $M_3(P, Q) + M_2(P, Q)$ has a pair of principal points A, B , and so $M_3(P, Q)$ contains a singular component $M_{s,3}(R'', A, S'')$. But since

$$M_1(P, Q) + M_2(P, Q) \equiv M_3(P, Q) + M_2(P, Q) \equiv M(A, B),$$

those parts of $M_1(P, Q)$ and $M_3(P, Q)$, which are not contained in $M_2(P, Q)$, must be identical with each other and accordingly the singular aggregates $\mathfrak{A}_{A,1}$ and $\mathfrak{A}_{A,3}$ of the both components must be identical. Therefore by the property of singular sets we have

$$M_{s,1}(R, A, S) \equiv M_{s,3}(R'', A, S'').$$

Therefore, in all cases, if the set has no pair of biprincipal points, any two points of the aggregate \mathcal{T} are continuously connected in that aggregate, and so the aggregate is semi-continuous.

Definition 14. If, in a continuous set, all of its components having two points A, B as a pair of principal points are simple, then the set is called a continuous one with simple components in regard to (A, B) or briefly a simple continuous set with respect to (A, B) .

Theorem 146. In a simple continuous set with respect to a pair of its biprincipal points (A, B) , if two constituent components, both of the second kind, have a third common point C , then the two components have a common continuous component containing A or B .

Proof. Suppose that $M_1(A, B)$ and $M_2(A, B)$ had neither common continuous component containing A nor that containing B , then, since $M_1(A, C)$ does not contain B and also $M_2(B, C)$ not contain A , there would be a point P of $M_2(A, B)$ belonging neither to $M_1(A, C)$ nor to $M_2(B, C)$ in neighborhood of A ; similarly a point P' of $M_1(A, B)$ belonging to neither of them in the neighborhood of B . Now the component $M_1(A, C) + M_2(B, C)$ is a continuous one containing A, B , and so it contains a continuous component having A, B as a pair of principal points. Denote it by $M_3(A, B)$, then it is different from $M_1(A, B)$ and $M_2(A, B)$, since it contains neither P nor P' . Thus the set has at least three different components $M_1(A, B)$, $M_2(A, B)$, and $M_3(A, B)$, which contradicts the hypothesis that (A, B) is a pair of biprincipal points. *Q. E. D.*

Cor. In a simple continuous set with respect to a pair of its biprincipal points, if its constituent components, both of the second kind, have no common continuous component containing one of the biprincipal points, then they have no common point other than the biprincipal points.

Theorem 147. In an ordinary simple continuous set with respect to a pair of its biprincipal points (A, B) , if one of its constituent components is a Jordan curve, then the constituents components can have no common point other than those belonging to a common continuous component containing one of the biprincipal points.

Proof. Denote by M_A and M_B the common continuous components containing A and B respectively; and suppose that there were a common point C belonging neither to M_A nor to M_B , then

$$M_1(A, B) \equiv M_1(A, C) + M_1(C, B),$$

$$M_2(A, B) \equiv M_2(A, C) + M_2(C, B).$$

Since C is a point not belonging to M_A and M_B , $M_1(A, C)$ and

$M_2(A, C)$ are not identical with each other, nor the one is contained in the other; similarly the same may be said of $M_1(B, C)$ and $M_2(B, C)$. Denote by $M_1^*(A, C)$ the part of $M_1(A, C)$ not belonging to $M_2(A, C)$, and by $M_2^*(A, C)$ that of $M_2(A, C)$ not belonging to $M_1(A, C)$, and by $M^*(A, C)$ the common part of the both components $M_1(A, C)$ and $M_2(A, C)$, then

$$M_1(A, C) \equiv M_1^*(A, C) + M^*(A, C),$$

$$M_2(A, C) \equiv M_2^*(A, C) + M^*(A, C).$$

Similarly

$$M_1(B, C) \equiv M_1^*(B, C) + M^*(B, C),$$

$$M_2(B, C) \equiv M_2^*(B, C) + M^*(B, C).$$

Now since (A, B) is a pair of biprincipal points the following relations must hold.

$$(a) \quad \left\{ \begin{array}{l} (i) \quad M_1(A, C) \text{ contains } M_1^*(B, C), \text{ or} \\ (ii) \quad M_2(B, C) \text{ contains } M_2^*(A, C), \end{array} \right.$$

and

$$(b) \quad \left\{ \begin{array}{l} (iii) \quad M_2(A, C) \text{ contains } M_2^*(B, C), \text{ or} \\ (iv) \quad M_1(B, C) \text{ contains } M_1^*(A, C).^{(1)} \end{array} \right.$$

If we assume that $M_1(A, B)$ is a Jordan curve, then (i), (iv) cannot occur. Hence we have only to discuss the case in which (ii) and (iii) occur at the same time. In this case $M_2(A, C)$ contains the three components $M^*(A, C)$, $M_2^*(A, C)$, and $M_2^*(B, C)$; and $M_2(B, C)$ contains the three components $M^*(B, C)$, $M_2^*(B, C)$ and $M_2^*(A, C)$. Thus $M_2(A, C)$ and $M_2(B, C)$ have all points common, other than those belonging to $M^*(B, C)$ and $M^*(A, C)$. But all these common points of

(¹) Precisely speaking, from the fact that $M_1(A, C) + M_2(B, C)$ must contain $M_1(A, B)$ or $M_2(A, B)$, it follows that

$$(a) \quad \left\{ \begin{array}{l} (i)' \quad M_1(A, C) \text{ contains those points of } M_1(B, C) \text{ which are not contained in } M_2(B, C), \text{ or} \\ (ii)' \quad M_2(B, C) \text{ contains those points of } M_2(A, C) \text{ which are not contained in } M_1(A, C), \text{ or} \\ (iii)' \quad M_1(A, C) \text{ contains those points of } M_2(A, C) \text{ which are not contained in } M_2(B, C), \text{ or} \\ (iv)' \quad M_2(B, C) \text{ contains those points of } M_1(B, C) \text{ which are not contained in } M_1(A, C); \end{array} \right.$$

and from the fact that $M_2(A, C) + M_1(B, C)$ must contain $M_1(A, B)$ or $M_2(A, B)$, either of the similar four cases (b) occurs. But (i)' and (iv)'; (ii)' and (iii)' state the same thing, therefore we have only to consider (i)' and (ii)' of (a). Similarly for (b).

$M_2(A, C)$ and $M_2(B, C)$ are conjugate principal points of A or B with respect to $M_2(A, C)$ or $M_2(B, C)$, and so they are limiting points of $M^*(B, C)$ or those of $M^*(A, C)$. Accordingly they must also belong to $M_1(A, C) + M_1(B, C)$, which contains both $M^*(A, C)$ and $M^*(B, C)$. Hence it follows that $M_2(A, B)$ is wholly contained in $M_1(A, B)$, but this is impossible. Therefore the set cannot contain the common point C .

Theorem 148. In a simple continuous set with respect to a pair of its biprincipal points (A, B) , if its constituent components, both of the second kind, having a common continuous component containing A , but not that containing B , have other common points, then all these common points are conjugate principal ones of B with respect to the both components $M_1(B, C)$ and $M_2(B, C)$ at the same time, where C denotes any one of the above common points.

Proof. Suppose that the constituent components have a common point C not belonging to M_A , then

$$M_1(A, B) \equiv M_1(A, C) + M_1(C, B),$$

$$M_2(A, B) \equiv M_2(A, C) + M_2(C, B).$$

Now since (A, B) is a pair of biprincipal points of the set, the following relations must hold.

$$(a) \quad \begin{cases} (i) & M_1(A, C) \text{ contains } M_1^*(B, C), \text{ or} \\ (ii) & M_2(B, C) \text{ contains } M_2^*(A, C), \end{cases}$$

and

$$(b) \quad \begin{cases} (iii) & M_2(A, C) \text{ contains } M_2^*(B, C), \text{ or} \\ (iv) & M_1(B, C) \text{ contains } M_1^*(A, C). \end{cases} \quad (\text{see Theorem 147}).$$

But as $M_1(A, B)$ and $M_2(A, B)$ are of the second kind and they have no common continuous component containing B , so (i) and (iii) cannot occur. Thus we have only to discuss the case in which (ii) and (iv) occur at the same time.

In this case $M_2(B, C)$ contains that part of $M_2(A, C)$ which is not contained in $M_1(A, C)$, and accordingly it contains at least a point P_C of M_A . Now the component $M_2(B, P_C)$ must contain the point C , since, if otherwise, the sum of the components $M_2(B, P_C)$ and $M_2(A, P_C)$, that is, $M_2(A, B)$ would not contain C , contrary to the hypothesis. Thus

$$M_2(B, P_C) \equiv M_2(B, C).$$

Similarly so is for $M_1(B, C)$ and $M_1(B, P_C)$, namely

$$M_1(B, P_C) \equiv M_1(B, C).$$

Therefore the common point C is a conjugate principal point of B with respect to the both components $M_1(B, P_c)$ and $M_2(B, P_c)$ at the same time.

Next take any other common point C_K not belonging to M_A , then by the same reasoning as above, we may prove that

$$M_2(B, P_{c_K}) \equiv M_2(B, C_K),$$

$$M_1(B_{c_K}, P) \equiv M_1(B, C_K).$$

Now since $M_2(A, P_{c_K})$, a component of M_A , does not contain C , so its complementary component $M_2(B, P_{c_K})$ must contain it, and accordingly also it must contain $M_2(B, C) \equiv M_2(B, P_c)$; conversely in a similar manner we may prove that $M_2(B, P_c)$ contains $M_2(B, P_{c_K})$. Therefore

$$M_2(B, P_c) \equiv M_2(B, P_{c_K}) \equiv M_2(B, C) \equiv M_2(B, C_K),$$

and similarly

$$M_1(B, P_c) \equiv M_1(B, P_{c_K}) \equiv M_1(B, C) \equiv M_1(B, C_K).$$

Thus all common points $\{C\}$ are conjugate principal ones of B with respect to the components $M_1(B, C)$ and $M_2(B, C)$ at the same time.

Cor. 1. In a continuous set having the property stated in the previous theorem, $M_1(A, B) - M_1(B, C)$ and $M_2(A, B) - M_2(B, C)$ are common to the both constituent components, and with their limiting points they form a common continuous component containing A .

Cor. 2. Each of the constituent components of a continuous set having the property stated in the previous theorem has a continuous part free from the common points⁽¹⁾ of the constituent components, in the neighborhood of B .

Proof. In a sufficiently small sphere described with B as centre, there are continuous components of $M_1(B, C)$ and $M_2(B, C)$ which contain B , but none of its conjugate principal points. But, by the previous theorem, all common points other than M_A are conjugate principal points of B with respect to $M_1(B, C)$ and $M_2(B, C)$; therefore the above continuous components are free from the common points.

From Theorem 148 and its corollaries, we have the clear knowledge of the constitution of the set; namely it consists of the three parts

(α) $M_A \equiv \overline{M_1(A, B) - M_1(B, C)}^{(2)} \equiv \overline{M_2(A, B) - M_2(B, C)}$, a common continuous component containing A ;

(¹) Here by the common points are meant those common ones, other than the biprincipal points A, B .

(²) $\overline{M_1(A, B) - M_1(B, C)}$ denotes the sum of the set $M_1(A, B) - M_1(B, C)$ and its limiting points, and this component may or may not be equal to M_A .

(b) $M_1(B, C)$;

(c) $M_2(B, C)$;

$M_1(B, C)$ and $M_2(B, C)$ have no common points other than their principal points.

Definition 15. In a continuous set $M(A, B)$ of the third kind if A and B be simple principal points of its components $M(A, C)$ and $M(B, C)$ respectively, C being any non-principal point of the set, then the set is said to be a simple type.

Here we shall study some properties of the set of simple type, as they are used in the further discussion.

Theorem 149. In a set $M(A, B)$ of the third kind, if A and B be simple principal points of $M(A, C)$ and $M(B, C)$ respectively, then all conjugate principal points $\{A_K\}$ of B with respect to $M(A, B)$ are also simple principal points of $M(A, C)$.

The same is true of all conjugate principal points of A .

Proof. By Theorem 23, Cor., any point of $\{A_K\}$ is a conjugate principal point of C with respect to $M(A, C)$. Now suppose that A_K were a compound principal point of $M(A, C)$ and denote two of its conjugate principal points by C and C_1 , then at least one of the two points A_K, C_1 , would be a conjugate principal point of A (Theorem 3). So A would be a compound principal point of $M(A, C)$, contrary to the hypothesis. *Q.E.D.*

Cor. In a continuous set, if A and B , a pair of its principal points, be simple principal points of its components $M(A, C)$ and $M(B, C)$ respectively, then the same is true of any pair of principal points of the set.

Theorem 150. If a set of the third kind of simple type contains a singular component, then all points of the component are principal ones of the given set.

Proof. Denote the given set by $M(A, B)$ and the singular component by $M_S(P_1, P_2, P_3)$.

I. If the singular component contain one of A, B , (say A), then it can contain no non-principal point of the set $M(A, B)$. For, if so, A would have all points of the singular aggregate, to which the non-principal point C belongs, as its conjugate principal points with respect to $M(A, C) \equiv M_S(P_1, P_2, P_3)$, and so A would be a compound principal one of the component, contrary to the hypothesis. Thus in this case all points of the singular component are principal ones of the given set $M(A, B)$.

II. If the singular component contain none of A, B , then by

Theorem 33, all points of $M_s(P_1, P_2, P_3)$ are only principal points or only non-principal points. But here the latter case cannot occur, since then at least one of $M(A, P_1)$ and $M(B, P_1)$ would be of the third kind having A or B as a compound principal point (Theorem 51), contrary to the hypothesis. Thus in this case also $M_s(P_1, P_2, P_3)$ contains only principal points.

Theorem 151. In a continuous set of simple type, any continuous component having two non-principal points as a pair of principal ones is a Jordan curve.

Proof. In a continuous set of simple type, $M(A, B)$, any non-principal point is a perfectly separating point, for, if otherwise, at least one of A, B would be a compound principal point of the separated components, contrary to the hypothesis. Therefore if C, D be any two non-principal points of the set, $M(A, C)$ and $M(B, C)$ have no common point other than C , and so D is contained in only one of the above components, (say $M(A, C)$); and thus any component having C, D as a pair of principal points must be wholly contained in $M(A, C)$ and so we have

$$M(A, C) \equiv M(A, D) + M(C, D).$$

But since $M(C, D)$ and $M(B, C)$ have only one common point C so again we have

$$M(C, D) + M(B, C) \equiv M(B, D).$$

Thus the set $M(A, B)$ may be decomposed into the three parts,

$$M(B, C), \quad M(C, D), \quad M(D, A),$$

each of them having only one common point with the following. Hence we may at once conclude that

1. the set has only one component having (C, D) as a pair of principal points,
2. the component $M(C, D)$ consists of only non-principal points.

If this component $M(C, D)$ contained a component of the third kind, then, by Theorem 130, it would also contain a component of the third kind having C or D as its compound principal point. Call this component $M(C, R)$, then

$$M(C, R) + M(C, B) \equiv M(R, B),$$

and, by Theorem 54, B will be a compound principal point of this component $M(R, B)$, which contradicts the hypothesis that $M(A, B)$ is a set of simple type, R being a non-principal point of the set. Therefore

$M(C, D)$ has no component of the third kind, and so it is a Jordan curve.

Cor. In a continuous set of simple type, any two points of the aggregate of non-principal points are connected by only one Jordan curve in that aggregate.

Theorem 152. In a continuous set with a pair of biprincipal points (A, B) , if its constituent components, both of the third kind of simple type, have a third common point, then either it is a conjugate principal point of a biprincipal point with respect to the both constituent components at the same time, or else it belongs to a common continuous component containing one of the biprincipal points.

Proof. Denote by C the common non-principal point, then we have

$$M_1(A, B) \equiv M_1(A, C) + M_1(C, B)$$

$$M_2(A, B) \equiv M_2(A, C) + M_2(C, B).$$

Now since (A, B) is a pair of biprincipal points, the following relations must hold:

$$(a) \quad \begin{cases} (i) & M_1(A, C) \text{ contains } M_1^*(B, C), \text{ or} \\ (ii) & M_2(B, C) \text{ contains } M_2^*(A, C), \end{cases}$$

and

$$(b) \quad \begin{cases} (iii) & M_2(A, C) \text{ contains } M_2^*(B, C), \text{ or} \\ (iv) & M_1(B, C) \text{ contains } M_1^*(A, C). \end{cases}$$

But, as $M_1(A, B)$ and $M_2(A, B)$ are of the simple type, $M_1(A, C)$ and $M_1(B, C)$ have no common point other than C , and similarly so is for $M_2(A, C)$ and $M_2(B, C)$. Therefore the above relations hold only when

$$M_1(B, C) \equiv M_2(B, C) \quad \text{or} \quad M_1(A, C) \equiv M_2(A, C),$$

namely when C belongs to a common continuous component containing one of the biprincipal points.

Next suppose that the common point C is a conjugate principal point of A with respect to $M_1(A, B)$, but not to $M_2(A, B)$, then, if $M_1(B, C)$ is not identical with $M_2(B, C)$, the component $M_2(A, C) + M_1(C, B)$ will contain a third component $M_3(A, B)$, which is different from $M_2(A, B)$, owing to the fact that $M_2(A, C) + M_1(C, B)$ does not contain some points of $M_2(B, C)$. This component is also different from $M_1(A, B)$, owing to the fact that $M_2(A, C)$ does not contain some points of $M_1(A, C) - M_1(C, B)$ (Theorems 150, 40_a, and 49). Thus the set must contain at least three different components $M_1(A, B)$, $M_2(A, B)$, and $M_3(A, B)$, which is impossible. Therefore C must be a conjugate principal point of A with respect

to $M_1(A, B)$ and $M_2(A, B)$ at the same time, if it does not belong to a common continuous component containing B .

Theorem 153. If, in an ordinary simple continuous set with respect to a pair of biprincipal points (A, B) , one of its constituent components is a Jordan curve, then the set has only one or only two continuous components having any two points of the set as a pair of principal points.

Proof. By Theorem 147, the constituent components have no common point other than those belonging to common continuous components containing one of A, B . Denote these common components by M_A and M_B , and any two points of the given set $M^{(2)}(A, B)$ by C, D . To prove the theorem we have to distinguish the following cases.

I. Both of the two points C, D belong to M_A (or M_B) at the same time.

In this case, it is clear that there is only one component having C, D as a pair of principal points.

II. The point C belongs to one of M_A and M_B , and the point D to the other of them.

In this case, there are only two components having C, D as a pair of principal points, namely one as a component of $M_1(A, B)$ and the other as a component of $M_2(A, B)$.

III. The point C belongs to one of M_A and M_B (say M_A), and the point D is any non-common point of the constituent components.

When D belongs to $M_1(A, B)$, $M_1(A, B)$ has one and only one component having C, D as a pair of principal points; denote it by $M_1(C, D)$. Further the sum of $M_1(D, B)$ and $M_2(A, B)$ contains another component having C, D as a pair of principal points, and the number of this latter component also must be only one. For, suppose that there are two components $M_j(C, D)$ and $M_k(C, D)$, then each of them must contain at least one point of M_B ; denote this point of $M_j(C, D)$ by Q and that of $M_k(C, D)$ by R , then

$$M_j(C, D) \equiv M_1(D, Q) + M_2(Q, C),$$

$$M_k(C, D) \equiv M_1(D, R) + M_2(R, C).$$

If R be an interior point of $M_1(D, Q)$, we have

$$M_1(D, Q) \equiv M_1(D, R) + M_1(R, Q),$$

$$M_2(C, Q) \equiv M_2(C, R) + M_2(R, Q),$$

and moreover

$$M_1(R, Q) \equiv M_2(R, Q).$$

Therefore, when $M_1(R, Q)$ is contained in $M_1(D, R)$ or $M_2(C, R)$,

$$M_j(C, D) \equiv M_k(C, D);$$

and, if otherwise, $M_k(C, D)$ would be contained in $M_j(C, D)$ as a proper component of it, which is impossible. Hence in this case the two components $M_j(C, D)$ and $M_k(C, D)$ must be identical with each other.

If R be not an interior point of $M_1(D, Q)$ and accordingly Q is that of $M_1(D, R)$ ⁽¹⁾, the same conclusion is obtained by the same reasoning.

IV. The points C, D belong to $M_1(A, B)$, but neither to M_A nor to M_B .

This case may be treated in a manner similar to the case III.

V. The point C belongs to $M_1(A, B)$ and the point D to $M_2(A, B)$, but neither to M_A nor to M_B .

In this case, the set $M_1(C, A) + M_2(A, D)$ has a component having C, D as a pair of principal points, and it is similarly proved as in the case III that such a component is only one. Also the set $M_1(C, B) + M_2(B, D)$ has only one component having C, D as a pair of principal points, different from the former.

Thus our theorem is completely established.

Theorem 154. If, in a simple continuous set with respect to a pair of biprincipal points (A, B) , one of whose constituent components is a Jordan curve, its constituent components have a common point C not belonging to a common continuous component containing one of the biprincipal points, then the one of the components necessarily contains a singular component and consists of the three parts:

1. $M(A, P)$, a common continuous component containing A ;
2. $M(B, Q)$, a common continuous component containing B ;
3. $M_S(P, Q, C)$, a singular component having a singular system of three points P, Q, C .

Proof. Proceeding as in Theorem 147, we may deduce from the hypothesis, in this case also, that $M_2(A, C)$ and $M_2(B, C)$ have all points common, other than those belonging to $M^*(A, C)$ and $M^*(B, C)$. Denote by $\{C_A\}$ the aggregate of common points of $M_1(A, C)$ and $M_2(A, C)$, not belonging to M_A , and by $\{C_B\}$ that of common points of $M_1(B, C)$ and $M_2(B, C)$, not belonging to M_B ; then we have

$$M^*(A, C) \equiv M_A + \{C_A\},$$

$$M^*(B, C) \equiv M_B + \{C_B\}.$$

(1) Since one of the constituent components is a Jordan curve, this is always the case.

But the component $M_1(A, B)$ being a Jordan curve, its continuous components M_A, M_B are of the second kind, and so we may denote them by $M_1(A, P) = M_2(A, P)$ and $M_1(B, Q) \equiv M_2(B, Q)$, and accordingly we have

$$M_2(A, C) \equiv M_2(A, P) + M_2(P, C),$$

and

$$M_2(B, C) \equiv M_2(B, Q) + M_2(Q, C).$$

Now in the component $M_2(P, C) - M_2(A, P)$, we can always determine a point R , such that the component $M_2(P, R)$ contains none of $\{C_A\}$, since $\{C_A\}$ does not belong to $M_A \equiv M_2(A, P)$. Therefore all points of $M_2(P, R) - M_2(A, P)$ are common to $M_2(A, C)$ and $M_2(B, C)$, and moreover all points of $M_2(R, C)$ and accordingly all points of

$$M_2(R, C) + M_2(P, R) - M_2(A, P) \equiv M_2(C, P) - M_2(A, P)$$

are also common to them by the simplicity of the component $M_2(A, B)$. (Theorem 89).

But $M_2(A, P)$ being a Jordan curve, and the aggregate of common points of $M_2(A, C)$ and $M_2(B, C)$ being continuous (Theorem 89), from the above result it follows at once that all points of $M_2(C, P)$ are common to $M_2(A, C)$ and $M_2(B, C)$. Similarly all points of $M_2(C, Q)$, a component of $M_2(B, C)$, are also common to $M_2(A, C)$ and $M_2(B, C)$. Now $M_2(A, P)$ and $M_2(B, Q)$ cannot have a common point, since, if they had, then the sum of them, a proper component of $M_2(A, B)$, would contain $M_2(A, B)$ itself. Therefore P must be contained in $M_2(C, Q)$ and Q in $M_2(C, P)$, and thus we have

$$M_2(C, P) \equiv M_2(C, Q).$$

Further $M_2(P, Q)$, a component of $M_2(C, Q)$, must contain C , since, if otherwise, the sum of $M_2(A, P)$, $M_2(P, Q)$, and $M_2(Q, B)$ would not contain C , while it contains the component $M_2(A, B)$ itself. Thus again we have

$$M_2(C, P) \equiv M_2(C, Q) \equiv M_2(P, Q),$$

which shows that this component has a singular system of three points P, Q, C . Q.E.D.

Cor. 1. If there are many common points $\{C\}$ having the same property as the point C , then all of them belong to $M_s(P, Q, C)$.

Cor. 2. The aggregate of all common points of $M_1(A, C)$ and $M_1(B, C)$ forms a singular continuous set.

Theorem 155. If, in a simple continuous set with respect to a pair of biprincipal points (A, B) , its constituent components, both of the second

kind, have a common continuous component containing A and also that containing B , then the set has only one pair of biprincipal points.

Proof. Suppose that the set had another pair of biprincipal points C, D , and let us discuss how the supposition leads us to a contradiction.

I. The case in which one of the two points C, D coincides with one of the two points A, B .

Assume that C coincides with A , and D belongs to $M_1(A, B)$, then one of the two constituent components (say $M_1(C, D)$) is a component of $M_1(A, B)$. Take any point Q other than B in the common continuous component containing B , and consider the sum of $M_1(D, Q)$ and $M_2(C, Q)$. Since $M_1(A, B)$ and $M_2(A, B)$ are of the second kind, the above component $M_1(D, Q)$ and $M_2(C, Q)$, and accordingly their sum does not contain B . And this sum is a continuous set containing C, D ; so it has at least one component having C, D as a pair of principal points, denote it by $M'(C, D)$, (if there be many, take any one of them). If this be different from $M_1(C, D)$, then there will be two components $M_1(C, D)$ and $M'(C, D)$, whose sum does not contain B , which contradicts the hypothesis that (C, D) is a pair of biprincipal points. If $M'(C, D)$ be identical with $M_1(C, D)$, it is clear that the sum of $M_1(D, Q)$, $M_2(C, Q)$, and $M_1(Q, B)$ contains also the component $M_1(C, D)$ and only this, as the one having C, D as a pair of principal points. So the given set contains only one component having C, D as a pair of principal points, which again contradicts the hypothesis. Q.E.D.

II. The case in which both of C, D are different from A, B , and they belong to one of the constituent components, say to $M_1(A, B)$. In this case, if C be an interior point of $M_1(A, D)$, then $M_1(A, D)$ contains $M_1(A, C)$ and $M_1(C, D)$. Now consider the three components $M_1(A, C)$, $M_1(D, Q)$, and $M_2(A, Q)$, where Q is a point having the same property as in the case I. Then since none of these three components contains B , so also their sum; and this sum is a continuous set containing C, D . Therefore, by the same reasoning as in the case I, we can prove that in this case also a contradiction arises.

III. The case in which both of C, D are different from A, B , and each of them belongs to different constituent components (say C to $M_1(A, B)$, and D to $M_2(A, B)$).

In this case consider the two sums of $M_1(C, A)$ and $M_2(A, D)$, and $M_1(C, Q)$ and $M_2(Q, D)$, where Q is a point having the same property as in the case I. These two sums are the continuous components containing C, D , and so each of them has at least one component having

C, D as a pair of principal points. Now these components are different from or identical with each other, but from either of them a contradiction always arises as before.

Thus the set has only one pair of biprincipal points A, B .

Theorem 156. *If, in a simple continuous set with respect to a pair of biprincipal points (A, B) , its constituent components, both of the second kind, have a common continuous component containing A , but not that containing B , then the set has an infinite number of pairs of biprincipal points, all of these pairs having A as their common element.*

Proof. By Theorem 148, Cor., the set has a continuous component which contains B , but no other common point of the constituent components. Take any point R of this component, then (A, R) is a pair of biprincipal points of the set. For, if R be a point belonging to $M_1(A, B)$, then $M_1(A, B)$ has one and only one component $M_1(A, R)$; and the sum of $M_2(A, B)$ and $M_1(B, R)$ forms another component having A, R as a pair of principal points, since they have no common point other than B . Denote this component by $M_2(A, R)$, then it is clear that $M_1(A, R)$ is different from $M_2(A, R)$, and the sum of them forms the whole set. Thus we have here only to prove that there is no other component having A, R as a pair of principal points.

Suppose that there were a third component $M_3(A, R)$, then the two cases would be possible.

I. The case in which $M_3(A, R)$ does not contain B .

Since $M_3(A, R)$ is different from $M_1(A, R)$, it must contain points of the both constituent components, and so it must contain at least one common point C of them not belonging to M_1 , so that $M_3(A, R)$ contains $M(A, C)$ and $M(C, R)$. But as the common point C has the property proved in Theorem 148, there is only one component having C, R as a pair of principal points, namely $M_1(C, R)$; and moreover the sum of $M_1(C, R)$ and $M(A, C)$ contains $M_1(A, C)$ and $M_1(C, R)$, though $M(A, C)$ is not identical with $M_1(A, C)$. Therefore $M_3(A, R)$ will contain $M_1(A, R)$, which is a contradiction.

II. The case in which $M_3(A, R)$ contains B .

In this case $M_3(A, R)$ contains a component having A, B as a pair of principal points. Denote it by $M_3(A, B)$, then since there are only two components $M_1(A, B)$ and $M_2(A, B)$ having A, B as a pair of principal points, $M_3(A, B)$ must be identical with one of them. But it cannot be identical with $M_1(A, B)$, since then $M_3(A, B)$ and accordingly $M_3(A, R)$ would contain $M_1(A, R)$. Therefore

$$M_3(A, B) \equiv M_2(A, B).$$

Besides $M_3(A, B)$, $M_3(A, R)$ contains a component having B, R as a pair of principal points. Denote it by $M_3(B, R)$. If this component $M_3(B, R)$ be identical with $M_1(B, R)$, then $M_3(A, R)$ would contain $M_2(A, R)$, which is again absurd. Thus $M_3(B, R)$ must consist of points of the both components $M_1(A, B)$, and $M_2(A, B)$, and so contain a common point C_1 of them not belonging to M_1 . Thus

$$M_3(B, R) \equiv M(B, C_1) + M(C_1, R),$$

$$\text{but} \quad M(C_1, R) \equiv M_1(C_1, R),$$

$$\text{therefore} \quad M_3(A, R) \equiv M_2(A, B) + M(B, C_1) + M_1(C_1, R) \\ > M_1(A, R).$$

Namely $M_3(A, R)$ contains $M_1(A, R)$, which is again absurd.

Therefore (A, R) is a pair of biprincipal points of the set, and the point R being any point of a continuous component, it is clear that there are an infinite number of pairs of biprincipal points of the set.

Further, when we take any two points of the set, each of which is different from A , we may prove that they cannot form a pair of biprincipal points, in a similar manner to Theorem 155. Therefore the point A must be an element of any pair of biprincipal points.

Theorem 157. If, in a simple continuous set with respect to a pair of biprincipal points (A, B) , its constituent components, both of the second kind, have neither a common continuous component containing A nor that containing B , then any point of the set is a conjugate biprincipal point of A and B at the same time. Namely the set has an infinite number of pairs of biprincipal points.

Proof. By Theorem 146, Cor., the constituent components have no common point other than A, B in this case. Take any interior point C of one of the constituent components (say $M_1(A, B)$), then $M_1(A, B)$ has one and only one component $M_1(A, C)$. Further the sum of $M_2(A, B)$ and $M_1(B, C)$ forms another component having A, C as a pair of principal points, since they have only one common point B ; denote this by $M_2(A, C)$. It is clear that the sum of $M_1(A, C)$ and $M_2(A, C)$ is identical with the whole set, and it is also clear that there is no third component $M_3(A, C)$, since $M_1(A, B)$ and $M_2(A, B)$ have only two common points A, B . Therefore (A, C) is a pair of biprincipal points, and similarly (B, C) is also a pair of biprincipal points. *Q.E.D.*

Though, in the above theorem, it was proved that any point of the set, other than A, B , is a conjugate biprincipal point of A and B at the

same time, yet we cannot assert that any two distinct points of the set always form a pair of biprincipal points. Here we shall find the condition that a set may have any two points of it as a pair of biprincipal points, and see what continuous set has such a property.

Theorem 158. If a continuous set has any two distinct points of it as a pair of biprincipal points, then it cannot contain a continuous component of the third kind (ordinary or singular), and its constituent components can have no common point other than its biprincipal points, that is, the set must be a closed Jordan curve.

This theorem is nothing but the verbal alteration of Theorem 139, so we need not to prove it again.

Conversely, any two points of a closed Jordan curve always form a pair of biprincipal points of it. Hence the theorem:

Theorem 159. The necessary and sufficient condition that any two points of a continuous set should be a pair of biprincipal points of it is that the set should be a closed Jordan curve.

Or we may say that the closed Jordan curve is a continuous set of points, such that any two points of it form a pair of biprincipal points of it.

Having discussed thus far the continuous set with a pair of biprincipal points, whose constituent components are of the second kind, we shall proceed to consider the case in which the constituent components are of the third kind.

In this case, Theorem 146 and its corollary are no longer true, for there is a set with biprincipal points, whose constituent components have a common point C , though they have no common continuous component containing one of the biprincipal points.

In the following discussions, under a continuous set is meant an ordinary simple continuous set with respect to a pair of biprincipal points, unless is stated otherwise.

Theorem 160. If a continuous set with a pair of biprincipal points (A, B) , whose constituent components have no common continuous component containing one of the biprincipal points, have a third common point C , then one and only one of (A, C) and (B, C) is a pair of principal points of the two constituent components at the same time.

Proof. Consider the four components $M_1(A, C)$, $M_1(B, C)$, $M_2(A, C)$, and $M_2(B, C)$. Since $M_1(A, C) + M_2(B, C)$ is a continuous set containing A, B , it must contain $M_1(A, B)$ or $M_2(A, B)$; otherwise the given set would have three different components having A, B as a pair of principal points. Now in order that $M_1(A, C) + M_2(B, C)$ should contain

$M_1(A, B) \equiv M_1(A, C) + M_1(B, C)$, it is necessary that $M_1(A, C)$ contains B , for, in the neighborhood of B , $M_2(B, C)$ does not contain an aggregate of points, which belongs to $M_1(A, B)$, and which has B as its limiting point, so that the aggregate and accordingly the point B must belong to $M_1(A, C)$. Therefore

$$M_1(A, C) \equiv M_1(A, B).$$

Also in order that $M_1(A, C) + M_2(B, C)$ should contain $M_2(A, B)$, it is necessary that

$$M_2(B, C) \equiv M_2(A, B).$$

Thus at least one of the two relations must hold :

$$(a) \quad \begin{cases} (i) & M_1(A, C) \equiv M_1(A, B) \\ (ii) & M_2(B, C) \equiv M_2(A, B). \end{cases}$$

Similarly, in order that $M_2(A, C) + M_1(B, C)$ should contain $M_1(A, B)$ or $M_2(A, B)$, at least one of the two relations must hold :

$$(b) \quad \begin{cases} (iii) & M_1(B, C) \equiv M_1(A, B) \\ (iv) & M_2(A, C) \equiv M_2(A, B). \end{cases}$$

Now (a) and (b) must be true at the same time, thus we have four combinations of them. But the cases (i), (iii); and (ii), (iv) cannot coexist, since, if so, the set would be singular owing to the fact that (A, B) , (A, C) , and (B, C) then would be pairs of principal points of $M_1(A, B)$ or $M_2(A, B)$. Similarly any three of four cases of (i), (ii), (iii), (iv) cannot coexist by the same reason, from which the latter part of our theorem follows. And from the coexistence of the cases (i), (iv), and also that of the cases (ii), (iii), the first part of our theorem follows.

Cor. 1. In the above set, the common points of the constituent components are divided into two aggregates, such that all elements of one aggregate are conjugate principal points of A with respect to the both constituent components, while all elements of the other are those of B with respect to the same components. In other words, any two points, each taken from the different aggregates, form a pair of principal points of the both constituent components at the same time, while any two points of the same aggregate never form it.

Cor. 2. In the above set, any non-principal point of the constituent components cannot be a common point of the two constituent components.

Cor. 3. If we admit the singularity of the constituent components, the theorem may be stated thus: If a set with a pair of biprincipal points

(A, B) , whose constituent components have no common continuous component containing one of the biprincipal points, has a third common point C , then either A, B, C form a singular system of three points of a constituent component, or one of (A, C) and (B, C) is a pair of principal points of the two constituent components at the same time.

Cor. 4. In a set with a pair of biprincipal points (A, B) , whose constituent components have no common continuous component containing one of the biprincipal points, if one of the components is of the second kind, then they have no common point other than A, B .

This is an extension of Theorem 146 Cor..

Theorem 161. In a set with a pair of biprincipal points (A, B) , whose constituent components have no common continuous component containing one of the biprincipal points, any third common point C of the constituent components is a conjugate biprincipal point of one and only one of its biprincipal points A, B .

Proof. In the case where (i), (iv) in the previous theorem hold simultaneously, it is at once seen that (A, C) is a pair of biprincipal points, but (B, C) is not so. Similarly in the case where (ii), (iii) hold simultaneously, (B, C) is a pair of biprincipal points, but (A, C) is not so.

Theorem 162. In the above set, the two aggregates of common points of its constituent components have properties, such that i) any element of the one forms a pair of biprincipal points with any element of the other, while ii) any two elements of the same aggregate never form a pair of biprincipal points.

Proof. Denote the two aggregates by $\{A\}$ and $\{B\}$, then by the previous theorem any element B_k of the aggregate $\{B\}$ is a conjugate biprincipal point of A , but not that of B . Now take (A, B_k) as a new pair of biprincipal points, then its constituent components $M_1(A, B_k)$ and $M_2(A, B_k)$ are identical with $M_1(A, B)$ and $M_2(A, B)$ respectively (Theorem 160). Therefore two aggregates of common points of the new constituent components are also $\{A\}$ and $\{B\}$, and so any element A_j of $\{A\}$ is a conjugate biprincipal point of B_k while any element B_l of $\{B\}$ is not so. Thus our theorem is proved, A_j and B_k being any elements of the two aggregates $\{A\}$ and $\{B\}$ respectively, and B_k, B_l being those of the same aggregate $\{B\}$.

From Theorem 160, Cor. 1, and Theorem 162, we have the following interesting theorem.

Theorem. The common points of the constituent components are divided

into two aggregates, such that i) any two points, each taken from the different aggregates, always form a pair of principal points of the both constituent components, and at the same time form a pair of biprincipal points of the set, while ii) any two points of the same aggregate never form the pairs of points of the both kinds above mentioned.

Theorem 163. In the above set, non-principal point of its constituent components is a conjugate biprincipal point of A and B at the same time.

Proof. Take any non-principal point P of a constituent component (say $M_1(A, B)$), then $M_1(A, B)$ has one and only one component having A, P as a pair of principal points; denote it by $M_1(A, P)$.

In the sum of $M_2(A, B)$ and $M_1(B, P)$, any continuous component containing A, P contains at least a point B_k of the aggregate $\{B\}$, since A and P belong separately to $M_2(A, B)$ and $M_1(B, P)$. Therefore the component contains $M(A, B_k)$ and $M(B_k, P)$; but from the property of B_k it follows at once that

$$M(A, B_k) \equiv M_2(A, B_k) \equiv M_2(A, B)$$

and

$$M(B_k, P) \equiv M_1(B_k, P) \equiv M_1(B, P).$$

Thus any component containing A, P always contains the whole set $M_2(A, B) + M_1(B, P)$, and so (A, P) is a pair of principal points of the sum; denote this set by $M_2(A, P)$.

By the property of common points of the constituent components, it is clear that there is no third component $M_3(A, P)$. Also it is clear that the sum of $M_1(A, P)$ and $M_2(A, P)$ is identical with the whole set $M_1(A, B) + M_2(A, B)$. Thus (A, P) is a pair of biprincipal points of the set. So is another pair of points (B, P) . *Q.E.D.*

Theorem 164. In the above set, any non-principal point of one of the constituent components form a pair of biprincipal points with any non-principal point of the other.

Proof. Take any non-principal points P_1 and P_2 from $M_1(A, B)$ and $M_2(A, B)$ respectively, then the sum of $M_1(A, P_1)$ and $M_2(A, P_2)$ has (P_1, P_2) as a pair of its principal points, since the common points of them are conjugate principal points of P_1 and P_2 with respect to them. Therefore

$$M_1(A, P_1) + M_2(A, P_2) \equiv M_A(P_1, P_2).$$

Similarly we have

$$M_1(B, P_1) + M_2(B, P_2) \equiv M_B(P_1, P_2).$$

Now by the property of common points of $M_1(A, B)$ and $M_2(A, B)$,

it is clear that there is no third component having (P_1, P_2) as a pair of principal points. Therefore (P_1, P_2) is a pair of bprincipal points of the set.

In the above set, if we denote by \mathcal{T}_A and \mathcal{T}_B the two aggregates of common points of the constituent components, and by Φ_1 and Φ_2 those of non-principal points of the constituent components, then from Theorems 162, 163, and 164, we have the following important theorem.

Theorem 165. In a continuous set with a pair of bprincipal points, whose constituent components have no common continuous component containing one of the bprincipal points, the four aggregates \mathcal{T}_A , \mathcal{T}_B , Φ_1 , and Φ_2 have such a property, that any two points, each taken from the different aggregates, always form a pair of bprincipal points of the set.

Theorem 166. In the above set, if the constituent components are of the simple type, then any two points in the aggregate of all non-principal points of the constituent components always form a pair of bprincipal points.

Proof. First take any two non-principal points P, Q of a constituent component, say $M_1(A, B)$, then $M_1(A, B)$ has one and only one component having P, Q as a pair of principal points, denote it by $M_1(P, Q)$. Since $M_1(A, B)$ is of the simple type, one and only one of $M_1(A, P)$ and $M_1(B, P)$ contains Q ; suppose that $M_1(B, P)$ contains it, then again $M_1(B, Q)$ does not contain P by the same reason. Now consider the sum of $M_1(A, P)$, $M_2(A, B)$ and $M_1(B, Q)$, then we can prove that the sum has (P, Q) as a pair of principal points, and also that there is no third component having (P, Q) as a pair of principal points, in a similar manner to Theorem 163, using the property of common points $\{A\}$ and $\{B\}$. Thus (P, Q) forms a pair of bprincipal points of the set.

Secondly, when P is taken from $M_1(A, B)$ and Q from $M_2(A, B)$, the points P, Q form a pair of bprincipal points, and this is proved in exactly the same manner as in Theorem 164. Hence follows the validity of our theorem.

Cor. In this set, the four aggregates \mathcal{T}_A , \mathcal{T}_B , Φ_1 , and Φ_2 have such properties, that i) any two points, each taken from the different aggregates, always form a pair of bprincipal points of the set, while ii) any two points taken from the same aggregate always do not form a pair of bprincipal points in \mathcal{T}_A and \mathcal{T}_B ; and always do in Φ_1 and Φ_2 .

Theorem 167. In a set with a pair of bprincipal points (A, B) , whose constituent components have a common continuous component containing A , but not that containing B , if the components have another common point C , then either (A, C) is a pair of principal points of the both constituent

components at the same time, or else $M_1(B, C)$ and $M_2(B, C)$ are both of the third kind having B as a compound principal point.

In the latter case, $M_1(A, B) - M_1(B, C)^{(1)}$ and $M_2(A, B) - M_2(B, C)^{(1)}$ are common to the both constituent components, and are contained in the common continuous component containing A .

Proof. Consider the four components $M_1(A, C)$, $M_1(B, C)$, $M_2(A, C)$, and $M_2(B, C)$. Since $M_1(A, C) + M_2(B, C)$ and $M_2(A, C) + M_1(B, C)$ are continuous components containing A, B , they must contain $M_1(A, B)$ or $M_2(A, B)$, and accordingly the following relations must hold;

- (i) $M_1(A, C) \equiv M_1(A, B)$, or
- (ii) $M_2(B, C)$ contains at least the part $M_2^*(A, C)$ of $M_2(A, C)$, which is not contained in $M_1(A, C)$;

and

- (iii) $M_2(A, C) \equiv M_2(A, B)$, or
- (iv) $M_1(B, C)$ contains at least the part $M_1^*(A, C)$ of $M_1(A, C)$, which is not contained in $M_2(A, C)$.

I. In the case in which (i) and (iii) occur at the same time, (A, C) is a pair of principal points of $M_1(A, B)$ and $M_2(A, B)$ at the same time.

II. In the case in which (ii) and (iv) occur at the same time, by (ii), $M_2(A, C)$ and $M_2(B, C)$ have a common part $M_2^*(A, C)$, and the remaining part of $M_2(A, C)$ is common to the both components $M_2(A, C)$ and $M_1(A, C)$. Thus, if $M_2(B, C)$ were of the second kind, then $M_2(A, C)$ would be of the third kind, and so $M_2^*(A, C)$ would be the set of limiting points of $M_2(A, C) - M_2^*(A, C)$, which is also a part of $M_1(A, C)$. Therefore $M_2^*(A, C)$ would be contained in $M_1(A, C)$, but this contradicts the assertion of (ii). Hence it follows that $M_2(B, C)$ is of the third kind having B as a compound principal point. Similarly from (iv), it follows that $M_1(B, C)$ is also of the third kind having B as a compound principal point.

Now if we denote by $\{C^{(1)}\}$ the aggregate of common points of $M_1(A, C)$ and $M_1(B, C)$, then from (iv) $M_1(A, C) - \{C^{(1)}\} \equiv M_1(A, B) - M_1(B, C)$ is contained in $M_2(A, C)$, and similarly $M_2(A, C) - \{C^{(2)}\} \equiv M_2(A, B) - M_2(B, C)$ is contained in $M_1(A, C)$, and so these parts are common to $M_1(A, C)$ and $M_2(A, C)$. Hence, by Theorem 123,

$$\overline{M_2(A, B) - M_2(B, C)}, \quad \overline{M_1(A, B) - M_1(B, C)}$$

are contained in a common continuous component containing A .

⁽¹⁾ Sometimes it may happen that one or both of $M_1(A, B) - M_1(B, C)$ and $M_2(A, B) - M_2(B, C)$ contain no point, yet in the former case the theorem is still true.

III. (i) and (iv) cannot occur at the same time. For, by (i), $M_1(B, C)$ is the set of conjugate principal points of A with respect to $M_1(A, B)$, and therefore is the set of limiting points of $M_1(A, B) - M_1(B, C)$. By (iv), $M_1(A, C) - M_1(B, C) \equiv M_1(A, B) - M_1(B, C)$ is contained in $M_2(A, C)$. Hence all points of $M_1(A, B)$ are contained in $M_2(A, C)$, which is of course impossible. Therefore (i) and (iv) cannot co-exist.

Similarly (ii) and (iii) cannot occur at the same time.

If in this theorem we denote by $\{P\}$ the aggregate of conjugate principal points of B with respect to $M_1(B, C)$ and $M_2(B, C)$, we may state the following theorem.

Theorem 168. In the set defined in the previous theorem, all the common points C 's of the latter kind belong to the aggregate $\{P\}$; and also all the non-common points $\{N\}$ of the two components $M_1(A, C)$ and $M_2(A, C)$ belong to the same aggregate $\{P\}$.

Proof. If there were a common point D of the latter kind, not belonging to $\{P\}$, then $M_1(B, D)$ would contain no point of $\{P\}$, since, if so, D must belong to the common continuous component containing A , contrary to the nature of the point D . Therefore all points of $\{P\}$ and especially an element C of it must be contained in $M_1(A, B) - M_1(B, D)$, which forms a part of common continuous component containing A . But this contradicts the hypothesis that C is a common point not belonging to the common continuous component containing A . Thus the first part of the theorem is proved.

Further, by (ii), all points of the non-common part $M_2^*(A, C)$ of $M_1(A, C)$ and $M_2(A, C)$ are common to the both components $M_2(A, C)$ and $M_2(B, C)$; so they are conjugate principal points of A or B with respect to the above components. But some of them cannot be those of A with respect to $M_2(A, C)$, for, if all of them were so, they would be limiting points of $M_2(A, C) - M_2^*(A, C) < M_1(A, C)$ and accordingly be contained in $M_1(A, C)$, contrary to the hypothesis. Therefore, by Theorem 63a, all of them are conjugate principal points of B with respect to $M_2(B, C)$, so that they belong to $\{P\}$. Similarly so is for all points of another non-common part $M_1^*(A, C)$. Thus the second part of the theorem is also proved.

From Theorem 168 may be deduced the following theorem.

If in the set defined in Theorem 167, C denotes any common point of the latter kind stated in that theorem, then all common points of $M_1(B, C)$ and $M_2(B, C)$ and all non-common points of $M_1(A, C)$ and $M_2(A, C)$ are the principal points of $M_1(B, C)$ and $M_2(B, C)$ at the same time.

Theorem 169. In a set with a pair of biprincipal points (A, B) , whose constituent components have a common continuous component containing A , and also that containing B , if the components have another common point C , then $M_1(A, C)$ and $M_2(A, C)$ are both of the third kind having A as a compound principal point, or else $M_1(B, C)$ and $M_2(B, C)$ have the same property.

In the both cases $M_1(A, B) - M_1(A \text{ or } B, C)$, $M_2(A, B) - M_2(A \text{ or } B, C)$ are common part of $M_1(A, B)$ and $M_2(A, B)$, and they are contained in the common continuous component containing one of the biprincipal points.

Proof. As in Theorem 167, the four components $M_1(A, C)$, $M_1(B, C)$, $M_2(A, C)$, and $M_2(B, C)$ must satisfy the following conditions:

- (a) $\left\{ \begin{array}{l} \text{(i) } M_1(A, C) \text{ contains that part } M_1^*(B, C) \text{ of } M_1(B, C), \\ \text{which does not belong to } M_2(B, C); \text{ or} \\ \text{(ii) } M_2(B, C) \text{ contains that part } M_2^*(A, C) \text{ of } M_2(A, C), \\ \text{which does not belong to } M_1(A, C); \end{array} \right.$

and

- (b) $\left\{ \begin{array}{l} \text{(iii) } M_2(A, C) \text{ contains that part } M_2^*(B, C) \text{ of } M_2(B, C), \\ \text{which does not belong to } M_1(B, C); \text{ or} \\ \text{(iv) } M_1(B, C) \text{ contains that part } M_1^*(A, C) \text{ of } M_1(A, C), \\ \text{which does not belong to } M_2(A, C). \end{array} \right.$

But the combinations (i) (iv), or (ii) (iii) cannot occur; for, if so, one of the constituent components would be contained in the other⁽¹⁾.

When the combinations (i) (iii) or (ii) (iv) occur, we may prove that $M_1(A, C)$ and $M_2(A, C)$, or $M_1(B, C)$ and $M_2(B, C)$ have the property stated in the above theorem as in Theorem 167. Of course in these cases it may happen that (A, C) or (B, C) is a pair of principal points of the constituent components.

When we denote by $\{C\}$ the aggregate of all conjugate principal points of A with respect to $M_1(A, C)$ and $M_2(A, C)$; and by $\{C'\}$ that of B with respect to $M_1(B, C)$ and $M_2(B, C)$, we may have the following theorem.

Theorem 170. In the set defined in the previous theorem, any common point not belonging to common continuous component containing one of the biprincipal points must belong to either $\{C\}$ or $\{C'\}$, but never to both at the same time.

This may be proved in a similar manner to Theorem 168.

Cor. In a continuous set with a pair of biprincipal points (A, B) ,

(¹) This may be proved as in the last part of Theorem 167.

whose constituent components have a common continuous component containing A , and also that containing B , if these common components be identical with the aggregate of principal points of at least one of the constituent components, then the constituent components cannot have other common points.

Thus all common points of the constituent components, not belonging to the common continuous component containing one of the biprincipal points, are contained in the principal points of either $M_1(A, C)$ or $M_1(B, C')$, and may be divided into two aggregates $[C]$ and $[C']$. The set may have only one or both of them.

Theorem 171. If the set has two aggregates $[C]$ and $[C']$, then all points not belonging to $M_1(C_k, C'_j)$ and $M_2(C_k, C'_j)$ are common points of the constituent components, and any pair of points (C_p, C'_j) , taken from $[C]$, $[C']$, is a pair of principal points of $M_1(C_k, C'_j)$ and $M_2(C_k, C'_j)$ at the same time.

Proof. By Theorem 169, all points not belong to $M_1(A, C_k)$ and $M_2(A, C_k)$ belong to the common continuous component containing B . So C'_j must belong to $M_1(A, C_k)$ and $M_2(A, C_k)$ at the same time. Therefore

$$M_1(A, C_k) \equiv M_1(A, C'_j) + M_1(C_k, C'_j),$$

$$M_2(A, C_k) \equiv M_2(A, C'_j) + M_2(C_k, C'_j),$$

and also

$$M_1(A, B) - M_1(B, C'_j) \equiv M_1^*(A, C'_j),$$

$$M_2(A, B) - M_2(B, C'_j) \equiv M_2^*(A, C'_j),$$

where $M_m^*(A, C'_j)$ denotes that part of $M_m(A, C'_j)$, which does not belong to $M_m(B, C'_j)$ ($m=1, 2$). But by Theorem 169, we have

$$\overline{M_1^*(A, C'_j)} \equiv \overline{M_2^*(A, C'_j)}.$$

Therefore it follows that all points which are not common to the both constituent components are contained in $M_1(C_k, C'_j)$ and $M_2(C_k, C'_j)$. Thus the first part of the theorem is proved.

Next taking a point C_p from $[C]$, we shall prove that (C_p, C'_j) is a pair of principal points of the component $M_1(C_k, C'_j)$. By hypothesis we have

$$M_1(A, C_p) \equiv M_1(A, C) \equiv M_1(A, C_k),$$

$$M_1(A, C_k) \equiv M_1(A, C'_j) + M_1(C_k, C'_j),$$

and

$$M_1(A, C_p) \equiv M_1(A, C'_j) + M_1(C_p, C'_j).$$

Now since C'_j is not a conjugate principal point of A with respect to $M_1(A, C_k)$, $M_1(A, C'_j)$ cannot contain C_k , and so its complementary component $M_1(C_p, C'_j)$ must contain it. Accordingly $M_1(C_p, C'_j)$ contains

$M_1(C_k, C'_j)$ by the simplicity of the constituent component. Similarly $M_1(C_k, C'_j)$ contains $M_1(C_p, C'_j)$, therefore

$$M_1(C_k, C'_j) \equiv M_1(C_p, C'_j).$$

In exactly the same manner, it may be proved that

$$M_2(C_k, C'_j) \equiv M_2(C_p, C'_j).$$

From these results, it is easily deduced that

$$M_1(C_k, C'_j) \equiv M_1(C_p, C'_q),$$

$$M_2(C_k, C'_j) \equiv M_2(C_p, C'_q). \quad \text{Q.E.D.}$$

Cor. The aggregates of non-principal points of two components $M_1(C, C')$ and $M_2(C, C')$ have no common point.

From Theorems 169, 170, 171, and its Cor., we have the clear knowledge of inner constitution of the continuous set stated in Theorem 169; namely its constituent components consist of

1. M_A , a common continuous part containing A ;
2. M_B , a common continuous part containing B ;
3. $[C]$, $[C']$, two closed aggregates of common points not belonging to M_A and M_B ;
4. $\{C\}_1$, $\{C'\}_1$, two continuous aggregates of principal points of $M_1(C, C')$;
5. $\{C\}_2$, $\{C'\}_2$, two continuous aggregates of principal points of $M_2(C, C')$;
6. $[N]_1$, $[N]_2$, two semi-continuous aggregates of non-principal points of $M_1(C, C')$ and $M_2(C, C')$.

Here the common parts of the two constituent components are two continuous parts M_A, M_B , and two closed aggregates $\{C\}$, $\{C'\}$; and the non-common parts are two semi-continuous parts $[N]_1$, $[N]_2$, and two aggregates of principal point $\{C\}_1 - [C]$, $\{C\}_2 - [C]$; $\{C'\}_1 - [C']$, $\{C'\}_2 - [C']$.

Here we shall study some interesting properties of the set defined in Theorem 170, Cor..

Theorem 172. In the set defined in Theorem 170, Cor., the aggregate of conjugate principal points of A (or B) with respect to one of the constituent components is wholly contained in the aggregate of the same property with respect to the other; and the two aggregates of non-principal points of the two constituent components have no common point.

Proof. Denote by $\mathcal{F}_{A,1}$ and $\mathcal{F}_{A,2}$ the aggregates of conjugate principal points of B with respect to the constituent components $M_1(A, B)$

and $M_2(A, B)$ respectively, and suppose that $\mathcal{F}_{A,1}$ is identical with the common continuous component containing A , then whether $\mathcal{F}_{A,2}$ is wholly contained in $\mathcal{F}_{A,1}$ or contains wholly $\mathcal{F}_{A,1}$ in it. That is, when $\mathcal{F}_{A,2}$ has certain points $\{A_2\}$, not contained in $\mathcal{F}_{A,1}$, all points of $\mathcal{F}_{A,1}$ belong to $\mathcal{F}_{A,2}$. For, if a point A_1 of $\mathcal{F}_{A,1}$ were a non-conjugate principal point of B with respect to $M_2(A, B)$, and so did not belong to $\mathcal{F}_{A,2}$, then $M_2(A_1, B)$ would contain no point of $\mathcal{F}_{A,2}$. But $\mathcal{F}_{A,1}$ being a common continuous component containing A , it contains a component $M_1(A, A_1) \equiv M_2(A, A_1)$, which of course does not contain $\{A_2\}$. Hence it follows that $M_2(A, B) \equiv M_2(B, A_1) + M_2(A_1, A)$ does not contain $\{A_2\}$, which is impossible. Thus the first part of our theorem is proved.

Next denote by Φ_1 and Φ_2 the aggregates of non-principal points of $M_1(A, B)$ and $M_2(A, B)$ respectively, then the part of Φ_1 contained in $M_2(A, B)$, if it exist, is contained in $\mathcal{F}_{A,2}$ or $\mathcal{F}_{B,2}$, but never in Φ_2 , as will be easily seen from the first part of the theorem, and also Theorem 170, Cor.. Similarly the part of Φ_2 , contained in $M_2(A, B)$, is contained in $\mathcal{F}_{A,1}$ or $\mathcal{F}_{B,1}$, but never in Φ_1 . Thus the latter part of the theorem is also proved.

From the properties above obtained, we may prove the following theorem as in Theorems 163 and 164.

Theorem 173. If the constituent components of the above set be of the simple type, then the set may be divided into the four aggregates \mathcal{F}_A , \mathcal{F}_B , Φ'_1 , and Φ'_2 having the following properties:

- (i) All points of $\mathcal{F}_A \equiv \mathcal{F}_{A,1} + \mathcal{F}_{A,2}$ are conjugate biprincipal points of B , but not of A ;
- (ii) all points of $\mathcal{F}_B \equiv \mathcal{F}_{B,1} + \mathcal{F}_{B,2}$ are conjugate biprincipal points of A , but not of B ;
- (iii) all points of $\Phi' \equiv \Phi'_1 + \Phi'_2$ (where Φ'_1, Φ'_2 denote the part of Φ_1, Φ_2 , not contained in $\mathcal{F}_A + \mathcal{F}_B$) are conjugate biprincipal points of A and B at the same time;

and also

- (i) any point of \mathcal{F}_A forms a pair of biprincipal points with any point of \mathcal{F}_B ;
- (ii) any point of Φ'_1 forms a pair of biprincipal points with any point of Φ'_2 ;
- (iii) any point of $\Phi' \equiv \Phi'_1 + \Phi'_2$ forms a pair of biprincipal points with any point of $\mathcal{F} \equiv \mathcal{F}_A + \mathcal{F}_B$.

That is, any two points, taken from any two of the four aggregates $\mathcal{F}_A, \mathcal{F}_B, \Phi'_1, \Phi'_2$, always form a pair of biprincipal points of the set.

In the above set, if $\Psi_{A,1} \equiv \Psi_{A,2}$ and $\Psi_{B,1} \equiv \Psi_{B,2}$, then we have the following interesting theorem.

Theorem 174. The set above defined is divided into two aggregates Ψ_A and Ψ_B of principal points of the constituent components and two aggregates Φ_1 and Φ_2 of non-principal points of them, having the following properties:

- (i) The two aggregates Ψ_A, Ψ_B are continuous;
- (ii) the two aggregates Φ_1, Φ_2 are semi-continuous, and have Ψ_A and Ψ_B as the aggregates of their limiting points;
- (iii) the four aggregates $\Psi_A, \Psi_B, \Phi_1, \Phi_2$ have such a property, that any two points, each taken from different aggregates, always form a pair of biprincipal points of the set, while any two points taken from the same aggregate never form it in Ψ_A and Ψ_B , and always form it in Φ_1 and Φ_2 .

It is to be remarked that one part of the properties of this set resembles very much those of the ordinary set stated in Theorem 42_A, while the other part resembles those of the singular set stated in Theorem 12.

Here we shall find a condition that the aggregates of principal points of two constituent components may be identical with each other.

Theorem 175. In a continuous set with a pair of biprincipal points (A, B) , whose constituent components have a common continuous component containing B , but not that containing A , if its constituent components have another common point C , which is a non-principal point of them, then A is a simple (or compound) principal point of the both constituent components at the same time; and the aggregates of all the conjugate principal points of A with respect to the both components are identical with each other.

Proof. Suppose that one of the constituent components, say $M_1(A, B)$, is of the third kind, and has A as a compound principal point, then $M_1(A, C)$ cannot contain any conjugate principal point of A , and so all conjugate principal points belong to the common continuous component M_B containing B (Theorem 167). Now denote by $\{B\}$ the aggregates of all conjugate principal points of A with respect to $M_1(A, B)$. Then M_B must contain certain points other than $\{B\}$, since otherwise $M_1(A, C)$ would contain $\{B\}$; denote by $\{Q\}$ the aggregate of the above points. From this aggregate take a point Q_k and consider a component $M^*(B, Q_k)$, which are common to the both constituent components; then this component $M^*(B, Q_k)$ is the set of the third kind having Q_k as a compound principal point and all points of $\{B\}$ as conjugate principal points of Q_k (Theorem 23). But the sum of $M_2(A, Q_k)$ and $M^*(B, Q_k)$ is identical with the constituent component $M_2(A, B)$, and so any point B_j of $\{B\}$

is the conjugate principal point of A with respect to $M_2(A, B)$ (Theorem 54). Conversely, by the same reasoning, we may prove that any conjugate principal point of A with respect to $M_2(A, B)$ is also that of A with respect to $M_1(A, B)$, so that the aggregates of conjugate principal points of A with respect to the both components are identical with each other.

Thus if A be a compound principal point of one of the constituent components, so also it is that of the other, and its conjugate principal points with respect to the both components are the same. Hence it also follows that if A be a simple principal point of one of the constituent components, so also it is that of the other. Q.E.D.

Cor. In a continuous set with a pair of biprincipal points (A, B) , whose constituent components have a common continuous component containing A and also that containing B , if each of the two aggregates of common points having the property stated in Theorem 171 contains a non-principal point of the both constituent components, then the aggregates of principal points of the both constituent components are identical with each other.

After having discussed the constitution of the continuous sets, which have a pair of biprincipal points and at least one of whose constituent components is of the third kind, we proceed to prove the following important theorem by the properties of the sets thus established.

Theorem 176. Any ordinary simple continuous set with respect to a pair of biprincipal points, at least one of whose constituent components is of the third kind, has an infinite number of pairs of biprincipal points.

Proof. To prove this we shall distinguish the four cases.

I. Two constituent components $M_1(A, B)$ and $M_2(A, B)$ have no common point other than A, B .

In this case, when $M_1(A, B)$ is a set of the third kind having A as its compound principal point, any conjugate principal point C of A with respect to $M_1(A, B)$ is a conjugate biprincipal point of A . For, to begin with,

$$M_1(A, C) \equiv M_1(A, B),$$

and since $M_1(A, B)$ is an ordinary set, $M_1(B, C)$ does not contain A ; so the two components $M_2(A, B)$ and $M_1(B, C)$ have no common point other than B , and accordingly their sum has (A, C) as a pair of principal points. Denote this component by $M_2(A, C)$, then it is clear that it is different from $M_1(A, C)$, and the sum of these components forms the whole set. It is also clear that there is no third component having (A, C) as a pair of principal points, owing to the fact that the constituent components have no common point other than A, B . Therefore (A, C) is a pair of biprincipal points of the set. And by Definition 7, the number

of conjugate principal points of A is infinite, so our theorem is proved.

Remark. In this case, even when the constituent components are singular, this theorem is still true. For, if we take C , so near to B that $M_1(B, C)$ does not contain A , the same reasoning as above still holds.

II. Two constituent components have common points other than A, B , yet they have no common continuous component containing one of the biprincipal points.

Denote by $\{C\}$ the aggregate of common points which are conjugate principal ones of A , and by $\{C'\}$ that of B (Theorem 160). Take any non-principal point P of one of the constituent components, say $M_1(A, B)$, then (A, P) is a pair of biprincipal points. In the first place, $M_1(A, B)$ has one and only one component having (A, P) as a pair of principal points; denote it by $M_1(A, P)$. Next the sum of $M_2(A, B)$ and $M_1(B, P)$ have also (A, P) as a pair of principal points. For, in the set $M_2(A, B) + M_1(B, P)$, consider any continuous component containing A, P , then since $M_1(B, P)$ does not contain A , and $M_2(A, B)$ not P , so any continuous component containing (A, P) must contain a common point C_k of the two constituent component and accordingly must contain the two components $M_2(A, C_k)$ and $M_1(C_k, P)$; but by Theorem 160,

$$M_1(C_k, P) \equiv M_1(B, P),$$

$$M_2(A, C_k) \equiv M_2(A, B).$$

Therefore any continuous component containing (A, P) always contains $M_2(A, B) + M_1(B, P)$. Thus (A, P) is a pair of principal points of it; denote this component by $M_2(A, P)$. The sum of $M_1(A, P)$ and $M_2(A, P)$ is clearly the original set itself, and it is also clear that there cannot be a third component having (A, P) as a pair of principal points. Hence (A, P) is a pair of biprincipal points, and by Theorem 30, the number of non-principal points is infinite, so our theorem is proved.

III. Two constituent components have a common continuous component containing one of its biprincipal points.

Denote by M_A the common continuous component containing A , one of the biprincipal points. By Theorem 167, other common points may be divided into two aggregates, namely, the one in which any element C of it is a conjugate principal point of A with respect to the both constituent components at the same time; and the other in which any element C' of it is a conjugate principal point of B with respect to

$M_1(B, C')$ and $M_2(B, C')$. Denote by $\{C\}$ the former aggregate, and by $\{C'\}$ the latter one. Here we have to distinguish the four cases.

(a). The set has common points of the former aggregate only.

In this case, from $M_1(A, B)$ take any point P which belongs neither to M_A nor to the set of conjugate principal points of A with respect to $M_1(A, B)$, and consider the component $M_1(B, P)$, then any non-principal point R of $M_1(B, P)$ is a conjugate biprincipal point of A . For, $M_1(B, R)$ contains neither M_A nor P , and so the sum of $M_2(A, B)$ and $M_1(B, R)$ is a continuous component having (A, R) as a pair of principal points, since the components have no common point other than $\{C\}$, every point of which is a conjugate principal point of A and R with respect to $M_2(A, B)$ and $M_1(B, R)$ respectively. Denote this component by $M_2(A, R)$, then it is clear that it is different from $M_1(A, R)$, and the sum of these two components is identical with the original set itself. It is also clear that there is no third component $M_3(A, R)$. Thus (A, R) is a pair of biprincipal points.

(b). The set has common points of the latter aggregate only.

In this case, take any non-principal point R of $M_1(B, C')$, then (A, R) is a pair of biprincipal points. Its proof runs as follows.

In the first place, since $M_2(A, B)$ and $M_1(B, R)$ have no common point other than B , the sum of them has (A, R) as a pair of principal points; denote it by $M_2(A, R)$. Then it is clear that this component is different from $M_1(A, R)$, and also that the sum of these two components is identical with the original set itself.

Further there cannot be a third component $M_3(A, R)$. For, any continuous component having (A, R) as a pair of principal points must contain some common points of the two constituent components, unless it is wholly contained in $M_1(A, B)$. If it contain a common point B , then it must consist of

$$M_1(R, B), \quad M_2(B, A),$$

$$\text{or} \quad M_1(R, B), \quad M_2(B, C'_k), \quad M_1(C'_k, A),$$

$$\text{or} \quad M_1(R, B), \quad M_2(B, C'_k), \quad M_1(C'_k, C'_j), \quad M_2(C'_j, A),$$

and similar ones. But all these sums contain the component $M_2(A, R) \equiv M_1(R, B) + M_2(B, A)$ by the property of C 's. Therefore there is no other component than $M_2(A, R)$.

If it contain a common point C'_k , but not B , then it must consist of

$$M_1(R, C'_k), \quad M_2(C'_k, A),$$

or $M_1(R, C'_k), \quad M_2(C'_k, C'_j), \quad M_1(C'_j, A),$
or
.....

But all these sums contain $M_1(A, R) \equiv M_1(R, C'_k) + M_1(C'_k, A)$. Therefore there is no other component than $M_1(A, R)$. Hence (A, R) is a pair of biprincipal points.

(c). The set has common points of the both aggregates.

In this case, if we take any non-principal point R of the component $M_1(C, C')$, then (A, R) is a pair of biprincipal points of the set.

(d). The set has no common point of the both aggregates.

In this case take any point R' having the same property as the point R in the case (a), then R is a conjugate biprincipal point of A . The proof may easily be effected in a similar manner to the case (a).

IV. Two constituent components have both of the common continuous components containing one of the biprincipal points.

It was proved that in this case those common points of the two constituent components, which do not belong to the common continuous component containing the biprincipal point, may be divided into two aggregates $[C]$ and $[C']$ having the property stated in Theorems 170 and 171. Moreover, by hypothesis, at least one of the constituent components is of the third kind, and so at least one of A, B is a compound principal point of that component. Suppose that A is a compound principal point of the component $M_1(A, B)$, then there is an infinite number of conjugate principal points of A with respect to $M_1(A, B)$ in the common continuous component containing B . Take any point R of them, then, in the following manner, we may prove that (A, R) is a pair of biprincipal points of the set.

In the first place, there are two different components having (A, R) as a pair of principal points, namely $M_1(A, R)$ and $M_2(A, R)$, each of them being a component of $M_1(A, B)$ and $M_2(A, B)$ respectively; and moreover the sum of these components is identical with the original set itself.

Next there is no third component having (A, R) as a pair of principal points. For, if there were a third component, then it would consist of points of the two constituent components and accordingly would contain common points of them. Thus it must consist of

$$M_2(R, C_k), \quad M_1(C_k, A),$$

or $M_2(R, C_k), \quad M_1(C_k, C_j), \quad M_2(C_j, A),$

or
 (A)
 or $M_2(R, B), M_1(B, C_k), M_2(C_k, A),$
 or

when the set has $[C]$ only as the aggregate of common points. But all these contain $M_1(A, R)$ or $M_2(A, R)$, so there cannot be a third component $M_3(A, R)$.

When the set has $[C']$ besides $[C]$, the same is still true. Instead of (A), here we have

$$M_2(R, C_k), M_1(C_k, C'_m), M_2(C'_m, A);$$

$$M_1(R, C_k), M_2(C_k, C'_m), M_1(C'_m, A);$$

and similar ones. But the one containing $M_2(C_k, C'_m)$ contains $M_2(A, R)$, and the one containing $M_1(C_k, C'_m)$ contains $M_1(A, R)$ by the properties of $[C]$ and $[C']$. Thus there cannot be a third component $M_3(A, R)$. Therefore (A, R) is a pair of biprincipal points of the set.

Further the case in which the set has neither $[C]$ nor $[C']$ may easily be treated in a similar manner. Thus in all cases our theorem is completely established.

Remark. It is interesting to note that, in the case III (c), any non-principal point R of $M_1(C, C')$ forms always a pair of biprincipal points with A , while, in the case IV, if C be a non-principal point of the both constituent components, any non-principal point R of $M_1(C, C')$ always does not form a pair of biprincipal points with A .

The latter part of the above proposition may be proved as follows.

By hypothesis, $M_1(A, C)$ and $M_2(A, C)$ do not contain B , and since R is a point of $M_1(C, C')$, so it is contained in $M_1(A, C)$. Now $M_1(A, C)$ contains a component $M_1(A, R)$ while $M_1(C, R) + M_2(A, C)$ contains another component $M_2(A, R)$. These components $M_1(A, R)$ and $M_2(A, R)$ are different from each other since $M_1(C, R) + M_2(A, C)$ surely does not contain certain points of $M_1(A, R)$ ⁽¹⁾. But the sum of $M_1(A, R)$ and

(1) If $M_1(C, R)$ contained all points of $M_1(C', R) - [C']$, then $M_1(C, R)$ would also contain $[C']$ since $[C']$ is the set of conjugate principal points of R with respect to $M_1(C', R)$ (Theorem 23, Cor.), but which contradicts the hypothesis that R is a non-principal point of $M_1(C, C')$. Therefore $M_1(C, R)$ does not contain certain points $\{P\}$ of $M_1(C', R) - [C']$, and accordingly those of $M_1(A, R)$. These points $\{P\}$ are not also contained in $M_2(A, C)$, since the two components $M_1(C, C')$ and $M_2(C, C')$ contain no common point other than those of $[C]$ and $[C']$. Hence $M_1(C, R) + M_2(A, C)$ does not contain certain points of $M_1(A, R)$.

$M_2(A, R)$ is a proper component of the original set as it does not contain B . Hence (A, R) cannot be a pair of biprincipal points of the set.

From Theorem 171 and its Cor., and also from the above remark, we have the following beautiful theorem.

Theorem. In the set defined in Theorem 171, (I) if a point of $[C]$ is a conjugate principal point of A with respect to a constituent component, then all points of $[C]$ are also conjugate principal points of A with respect to that component, and if a point of $[C]$ is a non-conjugate principal point of A , then all points of it are also non-conjugate principal points of A ; (II) if a point of $[C]$ is a conjugate principal point of A with respect to a constituent component, then all non-principal points of $M(C, C')$ are conjugate biprincipal points of A , and if a point of $[C]$ is a non-conjugate principal point of A with respect to the both constituent components, then all non-principal points of $M(C, C')$ are non-conjugate biprincipal points of A .

Classification of the Sets of the First Kind.

By the theorems above established, we may classify the ordinary continuous sets of the first kind into three categories:

- (i) that which has no pair of biprincipal points,
- (ii) that which has only one pair of biprincipal points,
- (iii) that which has an infinite number of pair of biprincipal points.

Of course under the continuous set here is meant the one, such that, when it has a pair of biprincipal points, its constituent components are simple.

Continuous set of points having a system of three points, every two of which form a pair of biprincipal points.

In the following discussions, under a continuous set is meant an ordinary simple one with respect to any one of the three pairs of biprincipal points, unless is stated otherwise.

Theorem 177. If any two of three points of a continuous set be a pair of biprincipal points, then any one of them cannot be a common point of constituent components having the other two as a pair of biprincipal points.

Proof. Let A, B, C be the three points having the said property, and suppose that C were a common point of $M_1(A, B)$ and $M_2(A, B)$, then

$$\begin{aligned} M_1(A, B) &\equiv M_1(A, C) + M_1(C, B), \\ M_2(A, B) &\equiv M_2(A, C) + M_2(C, B). \end{aligned} \quad (I)$$

(a). If $M_1(A, C)$ be identical with $M_2(A, C)$, there must be another component $M(A, C)$, which is complementary to $M_1(A, C)$ and so contains B , since $M_1(A, C)$ cannot contain B . But when any continuous component having (A, C) as a pair of principal points contains B , it must contain $M(A, B)$, which cannot be other than $M_1(A, B)$ or $M_2(A, B)$; accordingly it must contain $M_1(A, C)$, which is absurd. Thus in this case C cannot be a common point of $M_1(A, B)$ and $M_2(A, B)$.

(b). If $M_1(A, C)$ be different from $M_2(A, C)$, then at least one of them must contain B , because the sum of them is identical with the whole set. Therefore

$$(i) \quad M_1(A, C) \equiv M_1(A, B),$$

or

$$(ii) \quad M_2(A, C) \equiv M_2(A, B).$$

But by (i) $M_1(B, C)$ is the set of conjugate principal points of A with respect to $M_1(A, B)$, and therefore $M_2(B, C)$, the complementary set of $M_1(B, C)$, must contain $M_1(B, C)$ since $M_2(B, C)$ must contain at least $M_1(A, B) - M_1(B, C)$ (Theorem 42); but this is of course impossible.

The same may be said of the case (ii). Thus in the case (b) also the supposition that C were a common point of the two constituent components $M_1(A, B)$ and $M_2(A, B)$ leads us again to a contradiction. *Q.E.D.*

The previous theorem is stated for an ordinary set, but if we consider the continuous set in general we have the following interesting theorem.

Theorem 178. If any two of three points A, B, C be a pair of biprincipal points of a continuous set, and any one of them be a common point of the two constituent components having the other two as a pair of biprincipal points, then the two constituent components are both of the singular ones having the points A, B, C as their singular system of three points.

Proof. Since the two constituent components $M_1(A, B)$ and $M_2(A, B)$ have a common point C , each of them may be decomposed into two parts $M_1(A, C)$, $M_1(C, B)$; and $M_2(A, C)$, $M_2(C, B)$; that is,

$$\begin{aligned} M_1(A, B) &\equiv M_1(A, C) + M_1(C, B), \\ M_2(A, B) &\equiv M_2(A, C) + M_2(C, B). \end{aligned} \quad (I)$$

Of these decomposed components, if $M_1(A, C)$ be identical with $M_2(A, C)$, there must be another component $M(A, C)$, complementary to $M_1(A, C)$,

and this component $M(A, C)$ must contain B by hypothesis. But when any continuous component having (A, C) as a pair of principal points contains B , it must contain $M(A, B)$, which cannot be other than $M_1(A, B)$ or $M_2(A, B)$; accordingly $M(A, C)$ must contain $M_1(A, C)$, which is clearly absurd. Therefore $M_1(A, C)$ is not identical with $M_2(A, C)$, and so they are constituent components of the set; and accordingly by hypothesis they have a common point B . Hence it follows that $M_1(A, C)$, a component of $M_1(A, B)$, contains a component having (A, B) as a pair of principal points, which cannot be other than $M_1(A, B)$ itself. Thus we have

$$M_1(A, B) \equiv M_1(A, C); \quad (\text{II})$$

similarly

$$M_2(A, B) \equiv M_2(A, C).$$

The same reasoning may be applied to the other components $M_1(B, C)$ and $M_2(B, C)$, and the same result may be obtained. Thus

$$\begin{aligned} M_1(A, B) &\equiv M_1(B, C), \\ M_2(A, B) &\equiv M_2(B, C). \end{aligned} \quad (\text{III})$$

From (II) and (III), it follows that

$$\begin{aligned} M_1(A, B) &\equiv M_1(B, C) \equiv M_1(C, A) \equiv M_1^{\{3\}}(A, B, C), \\ M_2(A, B) &\equiv M_2(B, C) \equiv M_2(C, A) \equiv M_2^{\{3\}}(A, B, C). \end{aligned}$$

That is, the constituents are singular ones having the points A, B, C as their singular system of three points.

Theorem 179. *If any two of three points of a continuous set be conjugate biprincipal points of the other, then any two of them cannot be conjugate uniprincipal points of the other at the same time.*

Proof. Let the three points having the said property be denoted by A, B, C , and suppose that, if possible, B, C were conjugate principal points of A with respect to any one of $M_1(A, B)$ and $M_2(A, B)$, say $M_1(A, B)$, then

$$M_1(A, B) \equiv M_1(A, C).$$

Therefore $M_1(B, C)$, a component of $M_1(A, B)$, would be a set of conjugate principal points of A with respect to $M_1(A, B)$ (Theorem 19). And another component having (B, C) as a pair of principal points, which we shall denote by $M_2(B, C)$, would contain at least $M_1(A, B) - M_1(B, C)$. But then, since $M_1(B, C)$ is the set of limiting points of $M_1(A, B) - M_1(B, C)$, $M_2(B, C)$ must contain $M_1(B, C)$, which is clearly impossible. *Q.E.D.*

Theorem 180. If any two of three points of a continuous set form a pair of bprincipal points of the set, then any common point of two constituent components is a conjugate principal point of one of bprincipal points with respect to one or both of them.

In the former case, the constituent components have a common continuous component containing the common point and one of the bprincipal points.

Lemma. Let the three points having the said property be denoted by A, B, C , and P be a common point of $M_1(A, B)$ and $M_2(A, B)$; then P cannot be a common point of $M_1(A, C)$ and $M_1(B, C)$, components of $M_1(A, B)$.

Proof. Suppose that the point P were common to the both components $M_1(A, C)$ and $M_1(B, C)$, then $M_1(C, P)$ would also be common to them (Theorem 89). Now denote by $M_{1,A}$ and $M_{1,B}$ the remaining parts of $M_1(A, C)$ and $M_1(B, C)$ diminished by $M_1(C, P)$ respectively, then all points of $M_1(C, P)$ are limiting ones of $M_{1,A}$ or $M_{1,B}$. (i) If $M_1(B, P)$ be not identical with $M_2(B, P)$, the sum of $M_2(B, P)$ and $M_1(C, P)$ is identical with $M_2(B, C)^{(1)}$; and since the sum of $M_1(B, C)$ and $M_2(B, C)$ forms the whole set, $M_2(B, P)$ must contain all points of $M_{1,A}$ and moreover must have them as conjugate principal points of B . Therefore all points of $M_{1,A} + M_1(C, P) \equiv M_1(A, C)$ are limiting ones of $M_2(B, P) + M_1(B, C) - M_{1,A} - M_1(C, P) < M_2(A, C)$; whence follows that $M_2(A, C)$ must contain $M_1(A, C)$, contrary to the hypothesis. Thus in this case P is not a common point of $M_1(A, C)$ and $M_1(B, C)$. (ii) If $M_1(B, P)$ be identical with $M_2(B, P)$, then $M_1(A, P)$ cannot be identical with $M_2(A, P)$, and so by the same reasoning as above we arrive at the same conclusion. *Q.E.D.*

Proof of Theorem 180. Suppose that P belongs to $M_1(A, C)$ and it were not a conjugate principal point of B with respect to $M_1(A, B)$ and also to $M_2(A, B)$, then $M_1(C, P)$ and $M_2(B, P)$ and accordingly their

(1) Since $M_1(B, C)$ does not contain A , so its complementary component $M_2(B, C)$ must contain it and accordingly must contain also a component having (A, B) as a pair of principal points. But as it cannot be $M_1(A, B)$, it must be $M_2(A, B)$. Therefore $M_2(B, C)$ contains $M_2(B, P)$. Further, $M_2(B, C)$ contains a component having (A, C) as a pair of principal points. This component must be $M_1(A, C)$. For, if otherwise, $M_2(B, C)$ would contain all the points of the set, not contained in $M_1(A, C)$, and accordingly would contain $M_{1,B}$. Moreover $M_2(B, C)$ must contain all the points of the set, not contained in $M_1(B, C)$, so it contains $M_{1,A}$. Thus $M_2(B, C)$ contains $M_{1,B}$ and $M_{1,A}$, and accordingly their limiting points $\{C\}$, common to $M_1(A, C)$ and $M_1(B, C)$; which shows that $M_2(B, C)$ contains $M_1(B, C)$, contrary to the fact that (B, C) is a pair of bprincipal points of the set. Therefore $M_2(B, C)$ contains $M_2(B, P)$ and $M_1(P, C)$, so that

$$M_2(B, C) \equiv M_2(B, P) + M_1(P, C).$$

sum would not contain A ; but $M_1(C, P)$ and $M_2(B, P)$ being a component of $M_2(B, C)$ we must have

$$M_2(B, C) \equiv M_1(C, P) + M_2(B, P),$$

which shows that $M_2(B, C)$ does not contain A .

Further, by Theorem 179, $M_1(B, C)$ also does not contain A and therefore the sum of $M_1(B, C)$ and $M_2(B, C)$ cannot contain A ; but this is impossible since (B, C) is a pair of biprincipal points of the set. Therefore P must be a conjugate principal point of B with respect to at least one of the constituent components; the first part of the theorem is thus proved.

Next let us prove that when P is a conjugate principal point of B with respect to $M_2(A, B)$, but not to $M_1(A, B)$, $M_1(A, P)$ is wholly contained in $M_2(A, B)$; namely let us prove that $M_1(A, P)$ is a common continuous component of the two constituent components $M_1(A, B)$ and $M_2(A, B)$. Since, in this case, A and P are not contained in $M_1(B, C)$ (Lemma), so no point of $M_1(A, P)$ is also contained in $M_1(B, C)$ by the semi-continuity of the component $M_1(A, B) - M_1(B, C)$ (which contains A and P). Therefore all points of $M_1(A, P)$ and especially the point A is contained in $M_2(B, C)$; so $M_2(B, C)$ also contains a continuous component having (A, B) as a pair of principal points, which must be either $M_1(A, B)$ or $M_2(A, B)$. But as $M_1(A, B)$ contains $M_1(B, C)$, the component $M(A, B)$ contained in $M_2(B, C)$ must be $M_2(A, B)$. Thus $M_2(B, C)$ contains at least $M_2(A, B)$ and $M_1(A, P)$, and accordingly $M_1(A, P)$ and $M_2(A, P)$; whence follows that $M_1(A, P)$ is identical with $M_2(A, P)$, since $M_2(B, C)$ is simple. *Q.E.D.*

Theorem 181. If any two of the three points A, B, C of a continuous set be a pair of biprincipal points, then any point of $M(B, C)$ (that which does not contain A) is a conjugate biprincipal point of A . Of course the same is true of $M(A, B)$ or $M(A, C)$.

Proof. By Theorem 178, only one of $M_1(A, B)$ and $M_2(A, B)$ contains C . Suppose that the one which contains C is $M_2(A, B)$, then

$$M_2(A, B) \equiv M_2(A, C) + M_2(B, C). \quad (I)$$

Since, by Theorem 179, $M_2(A, C)$ does not contain B , its complementary component $M_1(A, C)$ must contain it, and accordingly also must contain $M(A, B)$ and $M(B, C)$. But there are only two components having (A, B) as a pair of principal points, and $M_2(A, B)$, one of them, contains $M_2(A, C)$, so we must have

$$M(A, B) \equiv M_1(A, B).$$

Now $M(B, C)$ cannot contain A , for, if so, then $M(B, C)$ would be identical with $M_1(A, C)$, and so A and B would be conjugate principal points of C at the same time, contrary to Theorem 179. Since there are only two components having (B, C) as a pair of principal points, and one of them must contain A , that which does not contain A must be identical with $M(B, C)$, so

$$M(B, C) \equiv M_2(B, C).$$

$$\text{Therefore} \quad M_1(A, C) \equiv M_1(A, B) + M_2(B, C). \quad (\text{II})$$

Take any point Q of $M_2(B, C)$, then

$$M_2(B, C) \equiv M_2(B, Q) + M_2(Q, C). \quad (\text{III})$$

Now by (II) and (III), the sum of $M_1(A, B)$ and $M_2(B, Q)$ is a component of the simple set $M_1(A, C)$; therefore its component having (A, Q) as a pair of principal points, which we denote by $M_1(A, Q)$, is identical with the sum itself or differs from it by $M_2(B, Q)$ at most, all points of which then being conjugate principal ones of C with respect to $M_2(B, C)$ (Theorem 104). The same may be said of the sum of $M_2(A, C)$ and $M_2(C, Q)$ and of its component $M_2(A, Q)$,

Of these two components $M_1(A, Q)$ and $M_2(A, Q)$, if $M_1(A, Q)$ be not identical with the sum $M_1(A, B) + M_2(B, Q)$, then $M_2(A, Q)$ is necessarily identical with the sum $M_2(A, C) + M_2(C, Q)$ and accordingly it contains $M_2(B, Q)$, since all points of $M_2(B, Q)$ and $M_2(C, Q)$ cannot be the principal points of $M_2(B, C)$ at the same time. Thus in all cases, the sum of $M_1(A, Q)$ and $M_2(A, Q)$ is identical with the whole set. Hence in order to prove that (A, Q) is a pair of biprincipal points we have only to show that there is no third component having (A, Q) as a pair of principal points.

Suppose that there were a third component $M_3(A, Q)$, then it must consist of points belonging to the both components $M_1(A, B)$ and $M_2(A, B)$, and accordingly must contain some common points of them. But, by Theorem 180, the common points may be divided into two aggregates $\{G_A\}$, $\{G_B\}$, the former of which consists of the conjugate principal points of B with respect to the both components, and of those belonging to a common continuous component containing A , while the latter consists of the conjugate principal points of A with respect to the both components, and of those belonging to a common continuous component containing B . If $M_3(A, Q)$ contain common points of $\{G_A\}$ only, then it would be identical with $M_2(A, Q)$, but this is absurd. If it contain

common points of both $\{G_A\}$ and $\{G_B\}$, then it would be identical with $M_1(A, Q)$, since it consists of points of $M_1(A, C)$ only, which is again absurd. Our theorem is thus established.

The converse of this theorem is not necessarily true, namely there is a continuous set, all points of whose component $M(B, C)$ are conjugate biprincipal points of A , yet (B, C) is not a pair of biprincipal points. For example, in the set defined by the equations

$$\left\{ \begin{array}{ll} y = \sin \frac{\pi}{x}, & -1 \leq x < 0 \\ y = (-1, +1), & x = 0 \\ y = \sin \frac{\pi}{x}, & 0 < x \leq 1 \\ y = (0, 2), & x = -1 \\ y = 2, & -1 < x < +1 \\ y = (2, 0), & x = 1 \end{array} \right.$$

if we take the three points $A(-1, 0)$, $B(0, +1)$, $C(0, -1)$, then (A, B) and (A, C) are pairs of biprincipal points, and all points of $M(B, C)$ are conjugate biprincipal points of A , yet (B, C) is not a pair of biprincipal points.

Definition 16. When a point of a continuous set with a pair of biprincipal points is not a conjugate biprincipal point of any other point of the set, the point is called a non-biprincipal point of the set.

Theorem 182. If a continuous set has three points, every two of which form a pair of biprincipal points of the set, then the set has no non-biprincipal point.

Proof. For, by the previous theorem, any point belonging to the component which has two of the three points A, B, C as a pair of principal points is a conjugate biprincipal point of the remaining one.

This property corresponds to that of the singular set with a pair of uniprincipal points.

It should be remarked that even in such a set any two points of the set are not necessarily a pair of biprincipal points, since it may contain a component of the third kind.

In the discussion of the continuous sets with a pair of uniprincipal points, we have already pointed out that, among the sets with which we are familiar, it is very difficult to find the one having a system of three points, every two of which form a pair of uniprincipal points; but on the contrary, in the continuous sets with a pair of biprincipal

points, there are many ordinary sets having the corresponding property. But, at the same time, as was already seen in the example of Theorem 181, there is also a continuous set having three points, such that one pair of points taken out of them is not that of biprincipal points, while the other two pairs of them are so. Thus a question naturally arises, if two pairs of points taken from three points be pairs of biprincipal points, then under what condition the remaining pair should also be a pair of biprincipal points? The following theorem gives the condition.

Theorem 183. The necessary and sufficient conditions that, in a continuous set having the two pairs of biprincipal points (A, B) , (A, C) , the two points B, C should also be a pair of biprincipal points are that

- (a) *B is not a conjugate principal point of A with respect to the both components $M_1(A, C)$ and $M_2(A, C)$;*
- (b) *when $M_k(A, C)$ contains B , and $M_k(A, B)$ and $M_j(A, C)$ have a common point D , $M_k(B, D)$ is a complementary set of $M_j(C, D)$ with respect to $M_k(A, B) + M_j(A, C)$ ($\begin{smallmatrix} k=1, 2 \\ j=1, 2 \end{smallmatrix} k \neq j$).*

Proof. (I) The conditions are necessary.

By Theorem 179, the condition (a) is necessary; and under this condition one and only one of $M_1(A, C)$ and $M_2(A, C)$ contains B (Theorem 177). Now let the one which contains B be $M_1(A, C)$, then it contains $M_1(B, C)$ and $M_1(A, B)$, and thus

$$M_1(A, C) \equiv M_1(A, B) + M_1(B, C). \quad \text{I.}$$

If this component $M_1(B, C)$ contain A , then $M_1(B, C)$ is identical with $M_1(A, C)$, and so $M_1(A, B)$ is the set of conjugate principal points of C with respect to $M_1(A, C)$. Hence it follows that (A, B) cannot be a pair of biprincipal points, which contradicts the hypothesis. Thus $M_1(B, C)$ cannot contain A .

Now in order that (B, C) may be a pair of biprincipal points, it is necessary that there is a second component having (B, C) as a pair of principal points and containing all points of the whole set, other than those of $M_1(B, C)$; denote this second component by $M_2(B, C)$. Since $M_1(B, C)$ does not contain A , so the above component $M_2(B, C)$ must contain it and accordingly must contain $M(A, B)$ and $M(A, C)$. But, since $M_1(A, C)$ contains $M_1(B, C)$, so

$$M(A, C) \equiv M_2(A, C).$$

Further, (i) if $M_1(A, B)$ and $M_1(B, C)$ have no common point other than B , it is clear that $M_1(A, B)$ must be wholly contained in $M_2(B, C)$; therefore in this case

$$M(A, B) \equiv M_1(A, B).$$

(ii) If $M_1(A, B)$ and $M_1(B, C)$ have common points other than B , then all these common points $\{B\}$ are principal points of at least one of $M_1(A, B)$ and $M_1(B, C)$ (Theorem 63_b). First suppose that all points of $\{B\}$ are conjugate principal ones of A with respect to $M_1(A, B)$, then since $M_1(A, B) - \{B\}$ is not contained in $M_1(B, C)$, so it and accordingly all points of $M_1(A, B)$ must be contained in $M_2(B, C)$; thus in this case also

$$M(A, B) \equiv M_1(A, B).$$

Next suppose that certain points of $\{B\}$ are not principal ones of $M_1(A, B)$, then all points of $\{B\}$ are conjugate principal ones of C with respect to $M_1(B, C)$ (Theorem 63_b), and so, since $M_2(A, B)$ must contain $M_1(B, C) - \{B\}$ by I, it must also contain $\{B\}$, accordingly $M_2(A, B)$ must contain $M_1(B, C)$. But, as $M_2(B, C)$ cannot contain $M_1(B, C)$, we have here again

$$M(A, B) \equiv M_1(A, B).$$

Therefore in all cases

$$M_2(B, C) \equiv M_2(A, C) + M_1(A, B). \quad \text{II.}$$

But in order that the above relation may hold, that is, (B, C) may be a pair of principal points of $M_2(A, C) + M_1(A, B)$, it is necessary that the condition (b) subsists (Theorem 106).

(II) The conditions are sufficient.

(a) When $M_1(A, C)$ contains B and accordingly contains $M_1(A, B)$ and $M_1(B, C)$, by the condition (b), $M_1(A, B) + M_2(A, C)$ is a continuous set having (B, C) as a pair of principal points; denote this component by $M_2(B, C)$; then we have

$$M_1(A, C) \equiv M_1(A, B) + M_1(B, C) \quad \text{I.}$$

$$\text{and} \quad M_2(B, C) \equiv M_1(A, B) + M_2(A, C). \quad \text{II.}$$

Therefore $M_1(B, C)$ is different from $M_2(B, C)$ and forms the whole set with it.

Thus to prove that (B, C) is a pair of biprincipal points, it is sufficient to show that there is no third component $M_3(B, C)$. From the relations I and II, we can easily deduce that

$$M_2(A, B) \equiv M_2(A, C) + M_1(B, C). \quad \text{III.}$$

If the above components $M_2(A, C)$ and $M_1(B, C)$ have common points other than C , denote them by $\{C\}$ and similarly those of $M_1(A, B)$

and $M_1(B, C)$ by $\{B\}$, then $M_1(B, C)$ and $M_2(B, C)$ cannot have common points other than those of $\{B\}$ and $\{C\}$. Now, since $M_1(A, B)$ and $M_1(B, C)$ are components of a simple set $M_1(A, C)$, the set of their common points $\{B\}$ forms one continuous set (Theorem 89). Similarly another set of common points $\{C\}$ also forms one continuous set. These continuous sets $\{B\}$ and $\{C\}$ can have no common point, since, if they had, then $\{B\} + \{C\}$ would be identical with $M_1(B, C)$, so that $M_1(B, C)$ would be contained in $M_2(B, C)$. Therefore $M_1(B, C)$ and $M_2(B, C)$ cannot have other common points besides those common continuous components containing one of B, C . Hence it follows at once that there cannot be a third component having B, C , as a pair of principal points. Thus in this case (B, C) is a pair of biprincipal points of the set.

(b) When $M_1(A, C)$ does not contain B , $M_2(A, C)$ contains it, therefore in this case also we arrive at the required result by proceeding in the same manner as in the case (a).

Theorem 184. When a simple continuous set $M(A, B)$ is divided into three parts M_A, M_B , and $\{M(A, B) - M_A - M_B\} \equiv M_C$, where M_A, M_B denote continuous components containing A and B respectively, there is always a point P in M_C , such that $M(A, P)$ contains no point of M_B , and $M(B, P)$ no point of M_A .

Proof. Taking a point Q in M_C , let us first consider the case in which a component $M(A, Q)$ contains some points $\{B_k\}$ of M_B . In this case all points of $\{B_k\}$ are conjugate principal points of A with respect to $M(A, Q)$, since, if otherwise, $M(A, B_k) + M(B, B_k)$ would be a continuous component containing A, B , but not Q . Denote the set of all conjugate principal points of A with respect to $M(A, Q)$ by $\{Q_i\}$, and any point of $M(A, Q) - \{Q_i\}$ by R ; then $M(A, R)$ does not contain Q and accordingly no point of M_B .

Now in the component $M(A, Q) - \{Q_i\}$, there are some points $\{S\}$, which are not contained in M_A ; for, if M_A contained all points of $M(A, Q) - \{Q_i\}$, so it would contain $\{Q_i\}$, and accordingly M_C , which contradicts the hypothesis that our set is divided into three parts. Taking a point S_m from $\{S\}$, consider a component $M(Q, S_m)$; if this component $M(Q, S_m)$ contain no point of M_A , then $M(B, Q)$ and accordingly $M(B, S_m)$ also contains no point of M_A , and $M(A, S_m)$ having no point of M_B , the point S_m has the said property.

If $M(Q, S_m)$ contain some points of M_A , then they are conjugate principal points of Q with respect to $M(Q, S_m)$. Denote all these conjugate principal points by $\{T\}$; then in $M(A, Q) - \{Q_i\}$ there must

be some points $\{P\}$ which are not contained in $M(A, S_m) + \{T\}$. For, if $M(A, S_m) + \{T\}$ contained all points of $M(A, Q) - \{Q_i\}$, since it is continuous, it would contain $\{Q_i\}$; and accordingly

$$M(A, S_m) + \{T\} \equiv M(A, Q) \equiv M(Q, S_m) + M(A, S_m).$$

But since $M(A, S_m)$ does not contain Q , so $\{T\}$ must contain it, namely the set of conjugate principal points of Q must contain Q , which is clearly impossible. Thus there exists an aggregate $\{P\}$.

Now taking P_n from $\{P\}$ consider a component $M(Q, P_n)$, then it contains no point of M_A , and since, of course, in this case $M(B, Q)$ can contain no point of M_A , so also their sum $M(B, Q) + M(Q, P_n)$ can not. Thus there is a component $M(B, P_n)$ which contains no point of M_A , and as was already proved, the component $M(A, P_n)$ contains no point of M_B . So the point P_n has the said property.

Next when $M(A, Q)$ contains no point of M_B , if $M(B, Q)$ also contain none of M_A , Q is the point having the said property. If $M(B, Q)$ contain some points of M_A , by proceeding as in the previous discussion, we shall also find a point having the said property. *Q. E. D.*

Remark. When $M(A, B)$ is a singular set, this theorem is not true.

Theorem 185. If a continuous set contains a system of three points, such that every two of them form a pair of biprincipal points of the set, then it has a system of four points having the same property.

Proof. Denote by A, B, C the three points of the above system, and suppose that $M_2(A, B)$ contains C , then, by Theorem 181, the following relations hold

$$M_2(A, B) \equiv M_2(A, C) + M_2(B, C), \quad \text{I.}$$

$$M_1(A, C) \equiv M_1(A, B) + M_2(B, C). \quad \text{II.}$$

Similarly we can prove that

$$M_1(B, C) \equiv M_1(A, B) + M_2(A, C)^{(1)}. \quad \text{III.}$$

So the whole set may be considered to consist of the three components $M_1(A, B)$, $M_2(A, C)$ and $M_2(B, C)$. Now denote by $\{C^{(2)}\}$

(1) Since $M_2(B, C)$ does not contain A , $M_1(B, C)$ must contain it and accordingly must contain $M(A, B)$ and $M(A, C)$. But $M_2(A, B)$ contains $M_2(B, C)$ by I, therefore

$$M(A, B) \equiv M_1(A, B).$$

Similarly

$$M(A, C) \equiv M_2(A, C),$$

whence follows that

$$M_1(B, C) \equiv M_1(A, B) + M_2(A, C).$$

the set of common points of $M_2(A, C)$ and $M_2(B, C)$; and by $\{B^{(2)(1)}\}$ that of $M_2(B, C)$ and $M_1(A, B)$; and by $\{A^{(2)(1)}\}$ that of $M_2(A, C)$ and $M_1(A, B)$. Since $\{C^{(2)}\}$ is a set of common points of $M_2(A, C)$ and $M_2(B, C)$, the components of a simple set $M_2(A, B)$, it is a continuous set containing C ; similarly $\{B^{(2)(1)}\}$ is a continuous set containing B .

Now it is clear that $M_2(B, C)$ has other points than those of $\{C^{(2)}\}$ and $\{B^{(2)(1)}\}$, since, if otherwise, $M_2(B, C)$ would wholly be contained in $M_2(A, C) + M_1(A, B) \equiv M_1(B, C)$ by III. Thus, by the previous theorem, $M_2(B, C)$ has a point P having the property stated in the same theorem. This point P forms a pair of biprincipal points with the point B . Its proof may be given as follows.

In the first place, $M_2(B, C)$ has a component having (B, P) as a pair of principal points; denote it by $M_1(B, P)$. Next, in the sum of $M_2(C, P)$, $M_2(A, C)$, and $M_1(A, B)$, which clearly does not contain certain points of $M_1(B, P)$ (Theorem 184), consider any continuous component \mathfrak{M} containing B, P . Since $M_2(C, P)$ and $M_1(A, B)$ have no common point, this component \mathfrak{M} must contain a common point of $M_2(A, C)$ and $M_1(A, B)$, namely a point A_k of $\{A^{(2)(1)}\}$, and so it must contain $M(A_k, B)$ and $M(A_k, P)$. But $M_2(A, C)$ and $M_2(C, P)$ being the components of a simple set $M_2(A, B)$, and P being a non-conjugate principal point of B with respect to $M_2(B, C)$, the sum of $M_2(A, C)$ and $M_2(C, P)$ is identical with $M_2(A, P)$ by Theorem 104; and moreover $M_1(A, B)$ and $M_2(A, P)$ have no other common point than those of $\{A^{(2)(1)}\}$, which has the property stated in Theorem 180. Hence it follows that $M(A_k, P)$ is a component of $M_2(A, P)$, and $M(A_k, B)$ that of $M_1(A, B)$. But again since $M_2(A_k, P)$ must contain C , we have

$$M_2(A_k, P) \equiv M_2(A_k, C) + M_2(P, C),$$

and also by the property of the simple set $M_1(B, C)$ and the relation III, we have

$$M_2(A_k, C) + M_1(A_k, B) \equiv M_1(B, C) \equiv M_1(A, B) + M_2(A, C).$$

Thus the component \mathfrak{M} contains all components $M_1(A, B)$, $M_2(A, C)$, $M_2(C, P)$; therefore $M_1(A, B) + M_2(A, C) + M_2(C, P)$ is a continuous component having (B, P) as a pair of principal points; denote this component by $M_2(B, P)$.

The sum of $M_1(B, P)$ and $M_2(B, P)$ is clearly identical with the whole set, and by what has just been proved, $M_2(P, C) + M_2(C, A) + M_1(A, B)$ has only one component $M_2(B, P)$ having (B, P) as a pair of principal points; and also $M_2(B, C)$ has only one component $M_1(B, P)$.

Therefore, if there were a third component $M_3(B, P)$, it would consist of a part of $M_1(B, P)$ and a part of $M_2(B, P)$. But their common points being contained in $\{B^{(2)(1)}\}$ and $\{P^{(2)(1)}\}$, it is easily seen that the third component $M_3(B, P)$ containing some of these common points cannot be other than $M_1(B, P)$ or $M_2(B, P)$ by the same reasoning as in Theorem 183. Thus (B, P) is a pair of biprincipal points of the set.

Similarly it may be proved that (C, P) is a pair of biprincipal points, and, by Theorem 181, (A, P) is also a pair of biprincipal points. Moreover, from the relations I, II, III, and the property of the simple set, it is easily seen that all constituent components $M_1(B, P), M_2(B, P); M_1(C, P), M_2(C, P);$ and $M_1(A, P), M_2(A, P)$ are also simple ones. So A, B, C, P form a system of four points having the said property.

Theorem 186. *If a continuous set contain a system of three points, such that every two of them form a pair of biprincipal points of the set, then it contains a system of n points having the same property, where n denotes any positive integer greater than 4.*

Proof. By the previous theorem, the set has a system of four points having the said property; denote the points by A_1, A_2, A_3, A_4 . In the component $M(A_r, A_s)$ not containing the other two points A_p, A_q , where p, q, r, s denote one of 1, 2, 3, 4, each different from the other, (it is clear by the previous theorem that there exists such a component), take a point A_5 having the property stated in Theorem 184, then, by the same reasoning as in the previous theorem, it may be proved that (A_r, A_5) and (A_s, A_5) are pairs of biprincipal points of the set; and by Theorem 181 (A_p, A_5) and (A_q, A_5) are also pairs of biprincipal points of the set. Therefore the five points A_1, A_2, A_3, A_4, A_5 form a system of points having the said property.

By repeating this process, we can form a system of n points having the same property, where n is any positive integer greater than 4.

This theorem corresponds to Theorem 12 of the continuous set with uniprincipal points. While it is very difficult to find such a continuous set in the case of uniprincipal points, it is very easy to have the sets of this kind in the case of biprincipal points, for example, any closed Jordan curve is one of this kind. There are also many continuous sets, which contain components of the third kind and yet have the above property.

**Comparison of properties of continuous set⁽¹⁾ having a
pair of biprincipal points with those of
continuous set having a pair
of uniprincipal points.**

When we compare the properties of the both sets, we find many interesting relations between them. The following are principal ones of them.

I. *If, in an ordinary continuous set having two pairs of uniprincipal points (A, B) , (A, C) , B, C be not a pair of uniprincipal points of the set, then all points of $M(B, C)$ are conjugate principal ones of A with respect to the set.*

But the corresponding theorem is not true in a continuous set having pairs of biprincipal points. Namely there is an ordinary continuous set which has (A, B) and (A, C) , but not (B, C) , as pairs of biprincipal points, and yet certain points (even all points except B, C) of whose component $M(B, C)$ are not conjugate biprincipal points of A . For example, consider two ordinary continuous sets which have only three common points A, B, C , and each of which has (A, B) and (A, C) as pairs of its principal points. In the sum of these sets, (A, B) and (A, C) are clearly pairs of biprincipal points of it; but (B, C) is not so, since the sum of $M_1(B, C)$ and $M_2(B, C)$ is a proper component of $M_1(A, B) + M_2(A, B)$. Now take any non-principal point P of $M_1(B, C)$, or $M_2(B, C)$, then the set has three different components having (A, P) as a pair of principal points, namely when P is a point of $M_1(B, C)$,

$$M_1(A, P) \equiv M_1(A, B),$$

$$M_2(A, P) \equiv M_2(A, B) + M_1(B, P),$$

$$M_3(A, P) \equiv M_2(A, B) + M_1(C, P);$$

and when P is a point of $M_2(B, C)$,

$$M_1(A, P) \equiv M_2(A, B),$$

$$M_2(A, P) \equiv M_1(A, B) + M_2(B, P),$$

$$M_3(A, P) \equiv M_1(A, B) + M_2(C, P).$$

Thus (A, P) is not a pair of biprincipal points.

On the contrary, *if in a continuous set with pairs of biprincipal points,*

(¹) Here under a continuous set having a pair of biprincipal points is meant an ordinary simple one with respect to that pair of biprincipal points, of which we have exclusively discussed hitherto.

it has a system of three points A, B, C , every two of which form a pair of biprincipal points, then any point of $M(B, C)$ is a conjugate biprincipal point of A (Theorem 181).

But the corresponding theorem is not necessarily true in the continuous set with pairs of uniprincipal points; namely there is a continuous set having a system of three points, every two of which form a pair of principal points, but some points of $M(B, C)$ are not conjugate principal points of A with respect to the set. This set we have called a singular one.

II. In continuous sets with pairs of biprincipal points, there is a set, any two points of which form always a pair of biprincipal points of the set (Theorem 159).

But in continuous sets with pairs of uniprincipal points, this is never the case; namely there is no such set, that any two points of it form a pair of uniprincipal points.

III. If a continuous set has three points, such that every two of them form a pair of biprincipal points, then the set has no non-biprincipal point (Theorem 182).

This theorem is also true in the set with pairs of uniprincipal points; namely, if a continuous set has three points, such that every two of them form a pair of uniprincipal points, then the set has no non-principal point (Theorem 9).

IV. If a continuous set has a system of three points, every two of which form a pair of biprincipal points, then the set has a system of n points having the same property, where n denotes any positive integer greater than 3 (Theorem 186). The number of such systems of n points is infinite.

This theorem has its correspondence in the continuous set with pairs of uniprincipal points (Theorem 12).

V. In a continuous set with pairs of uniprincipal points, the number of these pairs is only one or infinite. The same is also true of the set with pairs of biprincipal points.

Continuous set consisting of two constituent components $M_1(A, B)$ and $M_2(A, B)$ and yet having more than two components having (A, B) as a pair of principal points.

We have already discussed the continuous set which consists of two components $M_1(A, B)$ and $M_2(A, B)$, and which has no other component having (A, B) as a pair of principal points. Here we shall consider

some other sets of this kind, namely those which have also components $M_3(A, B)$, $M_4(A, B)$, other than the constituent ones. As in the previous discussion, here we shall treat the continuous set whose constituent components $M_1(A, B)$ and $M_2(A, B)$ are ordinary simple ones. But many theorems of this part are also true for any continuous set.

Theorem 187. *If, in a continuous set which consists of $M_1(A, B)$ and $M_2(A, B)$, the two constituent components have a finite or countably infinite number of common points, other than A, B , the set cannot have a third component $M_3(A, B)$, whose sum with $M_1(A, B)$ or $M_2(A, B)$ forms the given set.*

Proof. Denote by $\{C\}$ the set of common points of $M_1(A, B)$ and $M_2(A, B)$, and suppose that there were a third component $M_3(A, B)$ which forms the whole set with $M_1(A, B)$, then $M_3(A, B)$ must contain all the points of $M_2(A, B)$ except $\{C\}$. But since the number of points of $\{C\}$ is finite or countably infinite, any continuous component containing $M_2(A, B) - \{C\}$ must also contain $\{C\}$; so $M_3(A, B)$ must contain $M_2(A, B)$, which is absurd.

Theorem 188. *If, in a continuous set consisting of $M_1(A, B)$ and $M_2(A, B)$, none of common points, other than A, B , be a conjugate principal point of A or B with respect to $M_1(A, B)$ or $M_2(A, B)$, and the number of the common points be finite or countably infinite, then the set has two or more components having (A, B) as a pair of principal points, each of which is different from the constituent components.*

Proof. Take any common point R , then, by hypothesis, $M_1(A, R)$ does not contain B and accordingly its neighboring points. Thus $M_1(B, R)$ has a continuous component \mathfrak{M} containing B , but no point of $M_1(A, R)$. Further $M_2(B, R)$ cannot contain certain points of this component \mathfrak{M} , since the number of common points of $M_1(B, R)$ and $M_2(B, R)$ is at most countably infinite; so $M_1(A, R) + M_2(R, B)$ does not contain certain points of $M_1(B, R)$, and hence $M_3(A, B)$, contained in $M_1(A, R) + M_2(R, B)$, is different from $M_1(A, B)$. Similarly $M_3(A, B)$ is different from $M_2(A, B)$, owing to the fact that $M_1(A, R) + M_2(R, B)$ does not contain certain points of $M_2(A, R)$.

In exactly the same manner, we can prove that $M_4(A, B)$, contained in $M_2(A, R) + M_1(B, R)$, is also different from $M_1(A, B)$ and $M_2(A, B)$. Moreover $M_3(A, B)$ is different from $M_4(A, B)$. For, describe a sufficiently small sphere with A as centre, and consider a continuous component $M_4(A, S)$, wholly contained in this sphere. This component is of course a component of $M_2(A, R)$, and therefore it cannot be wholly

contained in $M_1(A, R)$. Nor it is contained in $M_2(B, R)$, if we take the radius of the sphere sufficiently small. So $M_4(A, B)$ contains certain points not contained in $M_3(A, B)$, and accordingly they are different from each other. *Q. E. D.*

More generally we may establish the following theorem. In this theorem we shall denote the set of common points of $M_p(A, R_1)$ and $M_q(R_1, R_2)$ by $\{C_{p,q}^{(1)}\}$; and that of $M_p(B, R_2)$ and $M_q(R_1, R_2)$ by $\{C_{p,q}^{(2)}\}$. ($p, q=1, 2, p \neq q$)

Theorem 189. If in a continuous set consisting of two components $M_1(A, B)$ and $M_2(A, B)$ having no common continuous component containing one of A, B ⁽¹⁾, $M_1(A, B)$ and $M_2(A, B)$ have two common points R_1, R_2 , such that i) R_1, R_2 are non-principal points of $M_1(A, B)$ and $M_2(A, B)$ at the same time; ii) $M_p(A, R_1)$ and $M_p(B, R_2)$ have no common point; iii) $M_q(C_{p,q}^{(1)}, C_{p,q}^{(2)})$ ⁽²⁾ is a complementary set of $M_p(A, C_{p,q}^{(1)})$ and $M_p(C_{p,q}^{(2)}, B)$ with respect to $M_p(A, R_1) + M_q(R_1, R_2)$ and $M_p(B, R_2) + M_q(R_1, R_2)$ respectively; then the set has the third and fourth components $M_3(A, B)$ and $M_4(A, B)$ whose sum is just the given set itself.

Proof. (a) First consider the case in which $M_1(R_1, R_2)$ and $M_2(R_1, R_2)$ are different from each other. In this case if we consider the sums of $M_1(A, R_1)$, $M_2(R_1, R_2)$, $M_1(R_2, B)$ and of $M_2(A, R_1)$, $M_1(R_1, R_2)$, $M_2(R_2, B)$, then, by Theorem 112, (A, B) is a pair of principal points of the first sum, and also of the second sum. Denote these sums by $M_3(A, B)$ and $M_4(A, B)$ respectively; then the sum of these components is clearly the given set itself.

Since R_1, R_2 are non-principal points of $M_1(A, B)$, $M_1(R_1, R_2)$ does not contain at least one of A, B and accordingly the neighboring points of it. The same may be said of $M_2(R_1, R_2)$. Thus $M_3(A, B)$ is not identical either with $M_2(A, B)$ or with $M_4(A, B)$; and also it is different from $M_1(A, B)$ since $M_1(R_1, R_2)$ and $M_2(R_1, R_2)$ are so. Similarly $M_4(A, B)$ is not identical with any one of $M_1(A, B)$, $M_2(A, B)$ and $M_3(A, B)$.

(b) Next consider the case in which $M_1(R_1, R_2)$ and $M_2(R_1, R_2)$ are identical with each other. In this case if we consider the sums of $M_1(A, R_1)$, $M_2(R_1, R_2)$, $M_2(R_2, B)$ and of $M_2(A, R_2)$, $M_1(R_1, R_2)$, $M_1(R_2, B)$, then, by Theorem 112, these sums are also continuous com-

(1) If $M_1(A, B)$ and $M_2(A, B)$ have no common continuous component containing one of A, B , they may have a finite or infinite number of common points.

(2) $M_q(C_{p,q}^{(1)}, C_{p,q}^{(2)})$ denotes a component contained in $M_q(R_1, R_2)$.

ponents having (A, B) as a pair of principal points. Denoting these sums by $M_3(A, B)$ and $M_4(A, B)$, we see at once that the sum of these components forms the whole set.

Since $M_1(A, B)$ and $M_2(A, B)$ have no common continuous component containing A , so $M_1(A, R_1)$ and $M_2(A, R_1)$ have different points from each other, in the neighborhood of A , and accordingly $M_3(A, B)$ is different from $M_2(A, B)$ and $M_4(A, B)$. Further since $M_1(R_2, B)$ and $M_2(R_2, B)$ have different points, in the neighborhood of B , so $M_3(A, B)$ is different from $M_1(A, B)$ also. Similarly $M_4(A, B)$ is different from any one of $M_1(A, B)$, $M_2(A, B)$ and $M_3(A, B)$. *Q. E. D.*

Considering the two points R_1, R_2 to be coincident in the case (b) of the previous theorem, we have the following theorem.

Theorem 190. If, in a continuous set whose constituent components $M_1(A, B)$ and $M_2(A, B)$ have no common continuous component containing one of A, B , the constituent components have a common point R , such that $M_p(A, C_{p,q})$ is a complementary set of $M_q(C_{p,q}, B)$ with respect to $M_p(A, R) + M_q(R, B)$, and R be a non-principal point of $M_1(A, B)$ and $M_2(A, B)$, then the set has the third and fourth components $M_3(A, B)$ and $M_4(A, B)$, whose sum is just the given set itself.

The proof may be established in a manner similar to the case (b) of the previous theorem.

Theorems 189 and 190 give a sufficient condition that a set consisting of $M_1(A, B)$ and $M_2(A, B)$ may have two components $M_3(A, B)$ and $M_4(A, B)$, whose sum is just the set itself; but to give its necessary and sufficient condition in general is difficult, since different cases occur according to the number of common points of $M_1(A, B)$ and $M_2(A, B)$. Here we shall give the necessary and sufficient conditions for the simple cases, in which the number of common points, other than A, B , is only one or two.

Theorem 191. The necessary and sufficient condition that a continuous set whose constituent components $M_1(A, B)$ and $M_2(A, B)$ have only one common point R , other than A, B , should have two components $M_3(A, B)$ and $M_4(A, B)$, such that the sum of them is just the original set itself, is that R should be a non-principal point of the both constituent components.

Proof. I. The condition is sufficient. For, since $M_1(A, R)$ and $M_2(B, R)$ have only one common point R , we have

$$M_1(A, R) + M_2(B, R) \equiv M_3(A, B),$$

and similarly

$$M_2(A, R) + M_1(B, R) \equiv M_4(A, B).$$

Now $M_1(A, R)$ and $M_1(B, R)$ have many non-common points since R is a non-principal point of $M_1(A, B)$; and also $M_2(A, R)$ and $M_2(B, R)$ have the same property. Therefore $M_3(A, B)$ and $M_4(A, B)$ are different from each other, and moreover they are also different from $M_1(A, B)$ and $M_2(A, B)$; and clearly the sum of them forms the whole set. Thus $M_3(A, B)$ and $M_4(A, B)$ have the required property.

II. The condition is necessary. For, if R be a conjugate principal point of A with respect to $M_1(A, B)$, then

$$M_1(A, R) \equiv M_1(A, B).$$

Now, by hypothesis, the common point of $M_1(A, B)$ and $M_2(A, B)$ is only one; so any component having (A, B) as a pair of principal points, which is different from $M_1(A, B)$ and $M_2(A, B)$, must consist of

$$(i) \quad M_1(A, R) \text{ and } M_2(B, R),$$

$$\text{or} \quad (ii) \quad M_2(A, R) \text{ and } M_1(B, R).$$

$$\text{But} \quad M_1(A, R) + M_2(B, R) \equiv M_1(A, B) + M_2(B, R),$$

thus the component of (i), which has (A, B) as a pair of principal points, is $M_1(A, B)$ itself. Hence there cannot be two components $M_3(A, B)$ and $M_4(A, B)$ having the required property. *Q. E. D.*

Let us now find the necessary and sufficient condition that a continuous set, whose constituent components $M_1(A, B)$ and $M_2(A, B)$ have only two common points R_1, R_2 , other than A, B , should have two components $M_3(A, B)$ and $M_4(A, B)$, whose sum is the original set itself. Here we have to distinguish three cases according as the two common points are principal ones of the constituent components or not.

(A) The case in which the two common points R_1, R_2 are principal points of $M_1(A, B)$.

Let us first consider the case in which R_1, R_2 form a pair of principal points. In this case, one of R_1, R_2 must be a conjugate principal point of A (say R_1), and the other that of B with respect to $M_1(A, B)$; and accordingly $M_1(A, R_2)$ is a set of conjugate principal points of B , and $M_1(B, R_1)$ that of A (Theorem 19). Now $M_3(A, B)$ must consist of

$$i) \quad M_1(A, R_2), M_2(R_2, R_1), \text{ and } M_1(R_1, B),$$

$$\text{or} \quad ii) \quad M_1(A, R_2) \text{ and } M_2(R_2, B),$$

$$\text{or} \quad iii) \quad M_2(A, R_1) \text{ and } M_1(R_1, B),$$

since $M_1(A, R_1), M_1(R_2, B)$ and $M_1(R_1, R_2)$ are all identical with $M_1(A, B)$. But in every one of the above three cases, $M_4(A, B)$ must contain at

least $[M_1(A, B) - M_1(A, R_2) - M_1(B, R_1)]$ in order that its sum with $M_3(A, B)$ should form the given set, and accordingly it must also contain $M_1(A, R_2)$ and $M_1(B, R_1)$ (Theorem 41). Namely $M_4(A, B)$ must contain $M_1(A, B)$ itself, and so cannot be other than $M_1(A, B)$. Thus the set cannot have two components $M_3(A, B)$ and $M_4(A, B)$ having the said property.

Next the case, in which both R_1 and R_2 are conjugate principal points of A (or B), may be similarly treated, and the same result may be obtained. Hence we have the theorem.

Theorem 192. When two common points R_1, R_2 are principal points of one of the two constituent components $M_1(A, B)$ and $M_2(A, B)$ of the set, the continuous set cannot have two components $M_3(A, B)$ and $M_4(A, B)$ whose sum is identical with the original set itself.

(B) The case in which one of the two common points R_1, R_2 is a non-principal point, and the other a principal point of $M_1(A, B)$.

Let R_2 be a non-principal point and R_1 be a conjugate principal point of A with respect to $M_1(A, B)$, then $M_1(A, R_2)$ contains neither B nor R_1 ; and moreover $M_1(B, R_2)$ is of the third kind having (B, R_1) as conjugate principal points of R_2 (Theorem 23). Now

$M_1(A, B) \equiv M_1(A, R_2) + M_1(R_2, B) \equiv M_1(A, R_2) + M_1(R_2, R_1) + M_1(R_1, B)$,
in which $M_1(R_1, B)$ is contained in $M_1(R_2, R_1)$; and

$$M_2(A, B) \equiv M_2(A, R_2) + M_2(R_2, R_1) + M_2(R_1, B).$$

Therefore only possible ways in which $M_3(A, B)$ may be constructed are as follows:

- i) by the combination of $M_1(A, R_2)$ and $M_2(R_2, B)$,
- ii) by the combination of $M_1(B, R_1)$ and $M_2(R_1, A)$,
- iii) by the combination of $M_1(B, R_2)$ and $M_2(R_2, A)$,
- iv) by the combination of $M_1(A, R_2)$, $M_2(R_2, R_1)$ and $M_1(R_1, B)$.

Other combinations may be reduced to the above or are impossible ones.

In order that i) $M_1(A, R_2) + M_2(R_2, B)$ may be a component $M_3(A, B)$ and be different from $M_1(A, B)$ and $M_2(A, B)$, it is necessary that $M_2(R_2, B)$ does not contain A . Further if, in this case, $M_2(A, R_2)$ contain R_1 , then any component having (A, B) as a pair of principal points, contained in the sum of $M_2(A, R_2)$ and $M_1(R_2, B)$, cannot contain $M_1(R_2, R_1) - M_1(R_1, B) - R_2$; and certain points of this latter component are not also contained in $M_3(A, B)$. Thus there is no fourth component having the required property. Accordingly it is necessary that $M_2(A, R_2)$ does not contain R_1 , and also of course B , in order that the set may have

the required component $M_4(A, B)$. Under these conditions it is clear that

$$M_1(A, R_2) + M_2(R_2, B) \equiv M_3(A, B),$$

$$M_2(A, R_2) + M_1(R_2, B) \equiv M_4(A, B),$$

and $M_3(A, B) + M_4(A, B) \equiv M_1(A, B) + M_2(A, B)$.

Thus in this case the required conditions are that R_2 is a non-principal point of $M_2(A, B)$, and $M_2(A, R_2)$ does not contain R_1 .

In the case ii) it is at once seen that there cannot be two components $M_3(A, B)$ and $M_4(A, B)$ having the required property.

In order that iii) $M_2(A, R_2) + M_1(R_2, B)$ may be a component $M_3(A, B)$ and be different from $M_1(A, B)$ and $M_2(A, B)$, it is necessary that $M_2(A, R_2)$ does not contain B . In this case, (a) if $M_2(A, R_2)$ does not contain R_1 , then

$$M_2(A, R_2) + M_1(R_2, B) \equiv M_3(A, B);$$

and further $M_1(A, R_2) + M_2(R_2, B) \equiv M_4(A, B)$,

provided that $M_2(R_2, B)$ does not contain A ; and in this case these two components $M_3(A, B)$ and $M_4(A, B)$ have the required property. (b) If $M_2(A, R_2)$ contain R_1 , then

$$M_2(A, R_1) + M_1(R_1, B) \equiv M_3(A, B),$$

and it reduces to the case ii), which have not the required components. Thus in this case also, the required conditions are that R_2 is a non-principal point of $M_2(A, B)$, and $M_2(A, R_2)$ does not contain R_1 . It is exactly the same as in the case i).

Lastly in the case iv), if $M_2(R_1, R_2)$ contain A , but not B , then it reduces to the case ii); and if it contain B , but not A , then it reduces to the case i); and if it contain A and B , or both of R_1, R_2 be conjugate principal points of one of A, B with respect to $M_2(A, B)$, then it reduces to the case stated in Theorem 192, in which the set has not the required components. Hence we have only to examine the case in which $M_2(R_1, R_2)$ contains neither A nor B , and at least one of R_1, R_2 is a non-principal point of $M_2(A, B)$. In this case we have

$$M_1(A, R_2) + M_2(R_1, R_2) + M_1(R_1, B) \equiv M_3(A, B).$$

Now if there be a fourth component $M_4(A, B)$, complementary to $M_3(A, B)$, then it must consist of $M_2(A, R_2)$, $M_1(R_1, R_2)$, and $M_2(R_1, B)$. But since $M_1(R_1, R_2)$ contains $M_1(R_1, B)$, so $M_2(A, R_2) + M_1(R_1, R_2)$ contains a component having (A, B) as a pair of principal points, and if this component be complementary to $M_3(A, B)$, it must contain that part of $M_2(R_1, B)$, which is not contained in $M_3(A, B)$. Namely $M_2(A, R_2)$

must contain the neighboring points of B in $M_2(R_1, B)$, and accordingly must contain the point B itself. Thus R_2 is a conjugate principal point of A with respect to $M_2(A, B)$, from which follows that $M_2(R_1, R_2)$ also contains B , contrary to the hypothesis. Therefore in this case there cannot be the required components. Hence we have the theorem.

Theorem 193. When one of the common points (say R_2) is a non-principal point of $M_1(A, B)$, and the other a conjugate principal point of A with respect to the same component, the necessary and sufficient conditions are that R_2 is also a non-principal point of $M_2(A, B)$ and R_1 is not contained in $M_2(A, R_2)$.

When R_1 is a conjugate principal point of A with respect to $M_2(A, B)$, the above conditions are of course satisfied, and we have the corollary.

Cor. If one of the two common points is a non-principal point of the both components $M_1(A, B)$, $M_2(A, B)$, and the other a conjugate principal point of A (or B) with respect to the both components above mentioned, then the set has two components $M_3(A, B)$ and $M_4(A, B)$ whose sum is just the original set itself.

If one of the two common points is a principal point of the both components at the same time, we have a very beautiful theorem concerning the necessary and sufficient condition that the set may have two components $M_3(A, B)$ and $M_4(A, B)$, whose sum is identical with the given set.

Theorem 194. When one of the two common points is a conjugate principal point of one of A, B with respect to the both components at the same time, the necessary and sufficient condition required is that the other of the common points is a non-principal point of the both components at the same time.

Proof. Let R_1 be a conjugate principal point of A with respect to the both components, then R_2 cannot be a conjugate principal point of B in order that the set may have two components $M_3(A, B)$ and $M_4(A, B)$ having the required property (Theorem 192). Moreover R_2 cannot be a conjugate principal point of A with respect to the both components at the same time; for, if R_2 were so, then any component having (A, B) as a pair of principal points would contain one of $M_1(A, R_1)$, $M_1(A, R_2)$, $M_2(A, R_1)$, $M_2(A, R_2)$, and accordingly $M_1(A, B)$ or $M_2(A, B)$.

Further if R_2 were a conjugate principal point of A with respect to one of the two components, say $M_1(A, B)$, and a non-principal point of the other, then only possible ways in which $M_3(A, B)$ may be constructed would be as follows:

- i) by the combination of $M_2(A, R_2)$, $M_1(R_1, R_2)$ and $M_2(R_1, B)$,

ii) by the combination of $M_2(A, R_2)$, $M_1(R_1, R_2)$ and $M_1(R_1, B)$. But in every one of these cases $M_1(A, B)$ must contain at least $[M_1(A, B) - \{B\}]$, where $\{B\}$ is the aggregate of conjugate principal points of A with respect to $M_1(A, B)$, and accordingly must contain $M_1(A, B)$ itself, which is absurd. Thus in order that the set may have the third and fourth components whose sum is just the original set itself, it is necessary that R_2 is a non-principal point of the both components.

Next the above condition is sufficient. For, since R_2 is a non-principal point and R_1 is a conjugate principal point of A , we have

$$\begin{aligned} M_1(A, B) &\equiv M_1(A, R_2) + M_1(R_2, R_1) + M_1(R_1, B), \\ M_2(A, B) &\equiv M_2(A, R_2) + M_2(R_2, R_1) + M_2(R_1, B), \end{aligned}$$

where $M_p(R_1, B)$ is contained in $M_p(R_2, R_1)$ [$p=1, 2$] and is the set of conjugate principal points of A . Therefore only possible ways in which $M_3(A, B)$ may be constructed are as follows:

i) by the combination of $M_1(A, R_2)$ and $M_2(R_2, B)$,

ii) by the combination of $M_2(A, R_2)$ and $M_1(R_2, B)$,

other combinations being reduced to the above, or impossible ones.

Now $M_1(A, R_2)$ and $M_2(R_2, B)$, and also $M_2(A, R_2)$ and $M_1(R_2, B)$ have no common point other than R_2 , consequently in i) we have

$$M_1(A, R_2) + M_2(R_2, B) \equiv M_3(A, B),$$

and

$$M_2(A, R_2) + M_1(R_2, B) \equiv M_4(A, B);$$

and in ii)

$$M_2(A, R_2) + M_1(R_2, B) \equiv M_3(A, B),$$

and

$$M_1(A, R_2) + M_2(R_2, B) \equiv M_4(A, B).$$

The sum of these components $M_3(A, B)$ and $M_4(A, B)$ is clearly identical with the original set itself, and they are different from each other and also from $M_1(A, B)$ and $M_2(A, B)$. Thus the set has two components $M_3(A, B)$ and $M_4(A, B)$ having the required property.

(C) The case in which the two common points R_1, R_2 are non-principal points of $M_1(A, B)$.

The case in which R_1, R_2 are principal points of $M_2(A, B)$ may be treated in the same manner as in the case (A); and the case in which one of R_1, R_2 is a non-principal point of $M_1(A, B)$ and the other a conjugate principal point of A or B with respect to the same component may be treated in the same manner as in the case (B), and the same result may be obtained. Thus we have only to consider the case in which R_1, R_2 are non-principal points of the two components $M_1(A, B)$ and $M_2(A, B)$ at the same time.

Let us divide the component $M_1(A, B)$ into parts as many as possible, such that every part is a continuous component having two of the four common points A, R_1, R_2, B as a pair of principal points, and it does not contain the common points other than its principal points. To obtain such a division, first consider the component $M_1(A, R_1)$, which does not contain B by hypothesis. If it contain R_2 , then R_2 may or may not be a conjugate principal point of A with respect to $M_1(A, R_1)$. Thus the three cases are to be distinguished :

i) $M_1(A, R_1)$ does not contain R_2 .

ii) $M_1(A, R_1)$ contains R_2 , but R_2 is not a conjugate principal point of A . In this case

$$M_1(A, R_1) \equiv M_1(A, R_2) + M_1(R_2, R_1),$$

and here $M_1(R_2, R_1)$ does not contain A , for, if it contained it, $M_1(R_2, R_1)$ would be identical with $M_1(A, R_1)$, and accordingly A, R_2 be conjugate principal points of R_1 with respect to $M_1(A, R_1)$, whence follows that they would also be conjugate principal points of B with respect to $M_1(A, B)$, contrary to the hypothesis. Therefore $M_1(R_2, R_1)$ contains neither A nor B .

iii) $M_1(A, R_1)$ contains R_2 , and has it as conjugate principal point of A , namely $M_1(A, R_1)$ is identical with $M_1(A, R_2)$.

In the case iii), (a) if the complementary component $M_1(B, R_1)$ of $M_1(A, R_1)$ with respect to $M_1(A, B)$ be of the third kind and has R_2 as a conjugate principal point of B , then

$$M_1(B, R_1) \equiv M_1(B, R_2).$$

(b) But if $M_1(B, R_1)$ be of the second kind, one of $M_1(B, R_1)$ and $M_1(B, R_2)$ cannot contain the common points⁽¹⁾ other than its principal points. Thus the case iii) may be again subdivided into three cases.

iii)_a $M_1(A, R_1) \equiv M_1(A, R_2)$, and $M_1(B, R_1) \equiv M_1(B, R_2)$.

iii)_b $M_1(A, R_1) \equiv M_1(A, R_2)$, and $M_1(B, R_1)$ contains neither A nor R_2 .

iii)_c $M_1(A, R_1) \equiv M_1(A, R_2)$, and $M_1(B, R_2)$ contains neither A nor R_1 .

Similarly the cases ii) and i) may be again subdivided as follows.

In the case ii), (a) if the complementary component $M_1(B, R_1)$ of $M_1(A, R_1)$ contain R_2 , then R_2 is a conjugate principal point of B with respect to $M_1(B, R_1)$, for, since $M_1(A, R_2)$ does not contain R_1 , so $M_1(B, R_2)$

(1) Here and hereafter by the common points are understood the four common points A, R_1, R_2, B of $M_1(A, B)$ and $M_2(A, B)$.

contains it, and conversely, by hypothesis, $M_1(B, R_1)$ contains R_2 , so that $M_1(B, R_2)$ is identical with $M_1(B, R_1)$. Thus in this case we have

$$\text{ii)}_a \quad M_1(A, B) \equiv M_1(A, R_2) + M_1(R_2, B).$$

(b) If $M_1(B, R_1)$ do not contain R_2 , then

$$\text{ii)}_b \quad M_1(A, B) \equiv M_1(A, R_2) + M_1(R_2, R_1) + M_1(R_1, B),$$

each of the above components containing no common point other than its principal points.

In the case i), since $M_1(A, R_1)$ does not contain R_2 , so $M_1(B, R_1)$ contains it. Treating this case as in the cases ii) and iii), we have

$$\text{i)}_a \quad M_1(A, B) \equiv M_1(A, R_1) + M_1(R_1, B),$$

where $M_1(R_1, B)$ is identical with $M_1(R_2, B)$, and $M_1(A, R_1)$ does not contain R_2 ; and

$$\text{i)}_b \quad M_1(A, B) \equiv M_1(A, R_1) + M_1(R_1, R_2) + M_1(R_2, B),$$

each of the above components containing no common point other than its principal points.

Thus all possible cases which may occur in dividing the component $M_1(A, B)$ into the ones having the required property are as follows.

$$\begin{aligned} [G_2^{(1)}] \left\{ \begin{array}{ll} \text{(I)}^{(1)} & M_1(A, R_1), M_1(R_1, R_2), M_1(R_2, B). \quad \text{(i)}_b \\ \text{(II)}^{(1)} & M_1(A, R_2), M_1(R_2, R_1), M_1(R_1, B). \quad \text{(ii)}_b \end{array} \right. \\ [G_3^{(1)}] \left\{ \begin{array}{ll} \text{(III)}^{(1)} & M_1(A, R_1) \equiv M_1(A, R_2), M_1(B, R_1) \equiv M_1(B, R_2). \quad \text{(iii)}_a \\ \text{(IV)}^{(1)} & M_1(A, R_1) \equiv M_1(A, R_2), M_1(B, R_1) \text{ or } M_1(B, R_2). \quad \text{(iii)}_b, \text{(iii)}_c \\ \text{(V)}^{(1)} & M_1(A, R_2) \text{ or } M_1(A, R_1), M_1(B, R_1) \equiv M_1(B, R_2). \quad \text{(ii)}_a, \text{(i)}_a \end{array} \right. \end{aligned}$$

Treating the component $M_2(A, B)$ in a similar manner, we have

$$\begin{aligned} [G_2^{(2)}] \left\{ \begin{array}{ll} \text{(I)}^{(2)} & M_2(A, R_1), M_2(R_1, R_2), M_2(R_2, B). \\ \text{(II)}^{(2)} & M_2(A, R_2), M_2(R_2, R_1), M_2(R_1, B). \end{array} \right. \\ [G_3^{(2)}] \left\{ \begin{array}{ll} \text{(III)}^{(2)} & M_2(A, R_1) \equiv M_2(A, R_2), M_2(B, R_1) \equiv M_2(B, R_2). \\ \text{(IV)}^{(2)} & M_2(A, R_1) \equiv M_2(A, R_2), M_2(B, R_1) \text{ or } M_2(B, R_2). \\ \text{(V)}^{(2)} & M_2(A, R_2) \text{ or } M_2(A, R_1), M_2(B, R_1) \equiv M_2(B, R_2). \end{array} \right. \end{aligned}$$

Now combine all cases concerning $M_1(A, B)$ to any one concerning $M_2(A, B)$, and examine whether they give the components $M_3(A, B)$ and $M_4(A, B)$ having the required property.

I. Combination of any one of the group $[G_2^{(1)}]$ with that of the group $[G_2^{(2)}]$.

In this case the set has two required components $M_3(A, B)$ and $M_4(A, B)$. For, as example, take $(\text{I})^{(1)}$ and $(\text{I})^{(2)}$, then each of

$$(a) \quad M_1(A, R_1) + M_2(R_1, R_2) + M_1(R_2, B),$$

$$(b) \quad M_2(A, R_1) + M_1(R_1, R_2) + M_2(R_2, B).$$

has (A, B) as a pair of principal points since every successive component of the sum has only one common point with its predecessor; and the sum of them is clearly the original set itself. Also it is clear that they are different from each other and moreover from $M_1(A, B)$ and $M_2(A, B)$. Therefore taking (a) and (b) as $M_3(A, B)$ and $M_4(A, B)$, we have the required components.

II. Combination of any one of the group $[G_3^{(1)}]$ with that of the group $[G_3^{(2)}]$.

In this case also the set has two required components $M_3(A, B)$ and $M_4(A, B)$. For, as example, take $(III)^{(1)}$ and $(III)^{(2)}$; and consider two components consisting of

$$(a) \quad M_2(A, R_1) \text{ and } M_1(R_1, B),$$

$$\text{and (b)} \quad M_1(A, R_1) \text{ and } M_2(R_1, B),$$

then they have the required property and may be taken as $M_3(A, B)$ and $M_4(A, B)$.

III. Combination of any one of the group $[G_2^{(1)}]$ with that of the group $[G_3^{(2)}]$; and combination of any one of the group $[G_2^{(2)}]$ with that of the group $[G_3^{(1)}]$.

In this case the set can not have two components $M_3(A, B)$ and $M_4(A, B)$ having the said property. For, as example, take $(I)^{(1)}$ and $(III)^{(2)}$, namely,

$$(I)^{(1)} \quad M_1(A, R_1), \quad M_1(R_1, R_2), \quad M_1(R_2, B),$$

$$(III)^{(2)} \quad M_2(A, R_1) \equiv M_2(A, R_2), \quad M_2(B, R_2) \equiv M_2(B, R_1),$$

then, as $M_3(A, B)$, we may only take

$$(a) \quad M_2(A, R_1) + M_1(R_2, B)$$

$$\text{or (b)} \quad M_2(B, R_1) + M_1(R_1, A).$$

But in the case (a), $M_4(A, B)$ must contain all the remaining points of the set, namely those of

$$M_1(A, R_1), \quad M_1(R_1, R_2) \text{ and } M_2(R_1, B);$$

nevertheless the sum of these three components cannot have (A, B) as a pair of principal points, since the sum of $M_1(A, R_1)$ and $M_2(R_1, B)$ has it. Thus in this case there cannot be the required components. Similarly in the case (b) we cannot have $M_4(A, B)$ having the said property. Thus we have the theorem.

Theorem 195. When R_1 and R_2 are both non-principal points of $M_1(A, B)$ and $M_2(A, B)$, the necessary and sufficient condition required is that $M_1(A, B)$ and $M_2(A, B)$ have or have not, at the same time, the proper components having R_1 and R_2 as conjugate principal points of A or B or both.

Theorem 196. If two sets of the second kind $M_1(A, B)$ and $M_2(A, B)$ have an aggregate of common points in the neighborhood of A , such that it has A as its limiting point, but is nowhere continuous, then the sum of the two sets has an infinite number of components having (A, B) as a pair of principal points.

Proof. In the aggregate of common points $\{E\}$, consider a sequence $E_1, E_2, \dots, E_n, \dots$, which has A as its limiting point and which has its element E_n as an interior point of $M_2(A, E_{n-1})$. Now taking a point E_m of it, consider the component $M_2(B, E_m)$; if this component $M_2(B, E_m)$ be entirely contained in the sum of $M_1(B, E_m)$ and $M_2(E_m, A)$, then again take a point E_{m+p} , such that $M_2(B, E_{m+p})$ is not entirely contained in the sum of $M_1(B, E_{m+p})$ and $M_2(E_{m+p}, A)$. (It is clear that such a point E_{m+p} exists in the set.) But if $M_2(B, E_m)$ be not entirely contained in the sum of $M_1(B, E_m)$ and $M_2(E_m, A)$, then consider that sum itself. In both cases, the above sum contains a component having (A, B) as a pair of principal points; denote it by $M_3(A, B)$. This component $M_3(A, B)$ is different from $M_1(A, B)$ and $M_2(A, B)$, since the above sum does not contain certain points of $M_1(A, E_{m+p})$ and also those of $M_2(B, E_{m+p})$. Thus we have a new component having (A, B) as a pair of principal points.

Now it is clear that there are an infinite number of points having the same property as that of the point E_{m+p} ; denote them by $(E_{m+p}, E_{m+p+q_1}, E_{m+p+q_2}, \dots)$ ($q_n < q_{n+1}$). In this aggregate, take a point E_{m+p+q_v} , such that $M_2(A, E_{m+p+q_v})$ and $M_1(B, E_{m+p})$ have no common point⁽¹⁾, then the component $M_4(A, B)$, contained in $M_1(B, E_{m+p+q_v}) + M_2(E_{m+p+q_v}, A)$, is different from $M_3(A, B)$. For, if they are identical with each other, they must consist of the points contained in the common parts of them and accordingly of the points contained in $M_1(B, E_{m+p}) + M_2(E_{m+p+q_v}, A)$ and of nowhere continuous aggregate $\{C\}$ of common points of $M_1(E_{m+p}, E_{m+p+q_v})$ and $M_2(E_{m+p}, E_{m+p+q_v})$. But $M_1(B, E_{m+p})$ and $M_2(E_{m+p+q_v}, A)$ having no common point, the sum

(1) This is always possible, since $M_1(A, B)$ is of the second kind and therefore we can describe a sufficiently small sphere with A as centre, such that it contains no point of $M_1(B, E_{m+p})$, while it contains a component $M_2(A, E_0)$ ($g > m+p$).

of them and $\{C\}$ cannot contain a continuous component containing A, B . Therefore $M_4(A, B)$ must be different from $M_3(A, B)$, and of course it is different from $M_1(A, B)$ and $M_2(A, B)$. Thus again we have a new component having (A, B) as a pair of principal points.

Further take a point E_{m+p+q_v+r} , such that $M_2(A, E_{m+p+q_v+r})$ and $M_1(B, E_{m+p+q_v})$ have no common point, then from this we may form a component $M_5(A, B)$, different from $M_1(A, B)$, $M_2(A, B)$, $M_3(A, B)$ and $M_4(A, B)$ as before. Proceeding in this way indefinitely we get an infinite number of components $M_1(A, B)$, $M_2(A, B)$, $\dots\dots\dots$, $M_n(A, B)$, $\dots\dots\dots$

Part II.

Continuous Set with a Pair of Triprincipal Points.

Definition 17. When a continuous set has two distinct points A, B , such that there are three and only three components having A, B as a pair of principal points, and the sum of them is just the set itself, these points A, B are called a pair of triprincipal points of the set.

Theorem 197. A continuous set with a pair of triprincipal points cannot have a pair of uniprincipal points.

Proof. Denote by $M_1(A, B)$, $M_2(A, B)$ and $M_3(A, B)$ the three constituent components of the set. It is clear that any pair of uniprincipal points (C, D) of the set, if it exist, cannot belong to one of the three constituent components. Assume that C belongs to $M_1(A, B)$ and D to $M_2(A, B)$, then $M_1(A, B) + M_2(A, B)$ is a continuous set containing C, D , and so it contains a continuous component having (C, D) as a pair of principal points (Theorem 1). But this component does not contain some points of $M_3(A, B)$, thus (C, D) can not be a pair of uniprincipal points of the set.

In investigating the properties of continuous set with a pair of triprincipal points, we shall distinguish two cases, namely (I) the case in which the three constituent components have no common point other than the triprincipal points, and (II) the case in which the three constituent components have common points other than the triprincipal points. We shall first discuss the case (I).

Case I.

Theorem 198. A continuous set with a pair of triprincipal points cannot have a pair of biprincipal points.

Proof. Denote by $M_1(A, B)$, $M_2(A, B)$ and $M_3(A, B)$ the three constituent components of the set. Taking any two points C, D of the set, we shall prove that C, D cannot be a pair of biprincipal points of the set.

I. The case in which both C and D belong to one of the constituent components, say to $M_1(A, B)$.

In this case $M_1(A, B)$ has a component having (C, D) as a pair of principal points; denote it by $M_1(C, D)$. If (C, D) were a pair of biprincipal points, then there would be a second component $M_2(C, D)$ which is a complementary of $M_1(C, D)$ with respect to the whole set. Thus $M_2(C, D)$ must contain at least $M_2(A, B)$, $M_3(A, B)$, C and D ; but the three constituent components having no common point other than A, B with one another, $M_2(C, D)$ must contain $M_1(A, C)$ or $M_1(A, D)$ beside the above components. Each of these components $M_1(A, C)$ and $M_1(A, D)$ cannot contain C and D at the same time, since, if so, $M_2(C, D)$ would contain $M_1(C, D)$ as its proper component. Therefore, when $M_2(C, D)$ contains $M_1(A, C)$, it must also contain the component $M_1(B, D)$, and when it contains $M_1(A, D)$, it also contains the component $M_1(B, C)$. So $M_2(C, D)$ must contain

(a) $M_2(A, B)$, $M_3(A, B)$, $M_1(A, C)$, and $M_1(B, D)$,

or (b) $M_2(A, B)$, $M_3(A, B)$, $M_1(A, D)$, and $M_1(B, C)$.

But in the case (a), $M_1(A, C) + M_2(A, B) + M_1(B, D)$ is a component having (C, D) as a pair of principal points; and $M_1(A, C) + M_3(A, B) + M_1(B, D)$ is another component having the same property; and each of them is a proper component of $M_2(C, D)$, which clearly contradicts the hypothesis that (C, D) is a pair of biprincipal points of the set.

In the case (b) the same reasoning leads us to the same contradiction. Therefore (C, D) cannot be a pair of biprincipal points of the set.

II. The case in which C belongs to one of the three components, say $M_1(A, B)$, and D to another, say $M_2(A, B)$.

In this case, if $M_1(A, B) + M_2(A, B)$ has two or more components having (C, D) as a pair of principal points, then clearly (C, D) cannot be a pair of biprincipal points. Thus we have only to examine the case in which the sum has only one component $M(C, D)$.

In the first place, this component $M(C, D)$ must be identical with the sum of $M_1(A, B)$ and $M_2(A, B)$. For, if $M(C, D)$ were a proper component of the sum, then at least one of the two components $M_1(A, C)$ and $M_2(A, D)$ could not contain B . And the one which does not con-

tain B (assume it to be $M_1(A, C)$) could not contain the points $\{Q\}$ of $M_1(A, B)$ in the neighborhood of B , and so

$$M_1(A, C) + M_2(A, D) \equiv M(C, D)$$

would not contain $\{Q\}$. Thus $M(C, D)$ is different from $M_1(B, C) + M_2(B, D)$, which surely forms another component having (C, D) as a pair of principal points. Therefore there are two different components having (C, D) as a pair of principal points in the sum of $M_1(A, B)$ and $M_2(A, B)$, contrary to the hypothesis. Accordingly $M(C, D)$ is identical with the sum of $M_1(A, B)$ and $M_2(A, B)$.

Next let us see what result will follow from the identity of $M(C, D)$ with the above sum. In this case

$$M_1(A, C) + M_2(A, D) \equiv M_1(A, B) + M_2(A, B) \equiv M_1(B, C) + M_2(B, D).$$

But since $M_1(A, B)$ and $M_2(A, B)$ have only two common points A, B , it follows

$$M_1(A, C) \equiv M_1(A, B) \equiv M_1(B, C),$$

and

$$M_2(A, D) \equiv M_2(A, B) \equiv M_2(B, D),$$

which shows that $M_1(A, B)$ and $M_2(A, B)$ are singular. From these results we may deduce that there cannot be a second component $M_2(C, D)$, complementary of $M(C, D)$ with respect to the whole set. For, if there were $M_2(C, D)$, then it would contain $M_3(A, B)$, $M_1(A, C) \equiv M_1(B, C)$ and $M_2(B, D) \equiv M_2(A, D)$; but since

$$M_1(A, C) \equiv M_1(A, B)$$

and

$$M_2(B, D) \equiv M_2(A, B),$$

$M_2(C, D)$ would contain $M(C, D)$ which is impossible. Therefore in this case also (C, D) cannot be a pair of biprincipal points of the set.

Theorem 199. In a set with a pair of triprincipal points (A, B) , the necessary and sufficient conditions that two interior points C, D of one constituent component $M_r(A, B)$ ($r=1, 2, 3$) should be a pair of triprincipal points are that $M_r(A, B)$ should have only one component having (C, D) as a pair of principal points, and $M_r(A, C)$ and $M_r(B, D)$ have no common point, where C is an interior point of $M_r(A, D)$.

Proof. I. The conditions are sufficient.

By hypothesis, $M_r(A, B)$ has one and only one component $M_r(C, D)$, and since $M_r(A, C)$ and $M_r(B, D)$ have no common point, $M_r(A, C) + M_r(B, D)$ does not contain certain points of $M_r(C, D)$. Therefore

$$M_r(C, A) + M_s(A, B) + M_r(B, D) \quad (s=1, 2, 3; s \neq r)$$

is a component having (C, D) as a pair of principal points and is different from $M_r(C, D)$. Thus there are three and only three components having (C, D) as a pair of principal points, and the sum of them is clearly the given set itself.

II. The conditions are necessary.

If $M_r(A, C)$ and $M_r(B, D)$ have common points, then the sum of them is identical with $M_r(A, B)$, and therefore contain $M_r(C, D)$. Now suppose that (C, D) is a pair of triprincipal points, then there must be a component $M(C, D)$ containing $M_s(A, B)$. Since $M_s(A, B)$ and $M_r(A, B)$ have no common point other than A, B , the component $M(C, D)$ must contain at least

$$M_s(A, B), \quad M_r(A, C), \quad M_r(B, D)$$

or

$$M_s(A, B), \quad M_r(A, D), \quad M_r(B, C),$$

but this is absurd owing to the fact that $M_r(A, C) + M_r(B, D)$ contains $M_r(C, D)$, and $M_r(A, D)$ also contains $M_r(C, D)$, C being an interior point of $M_r(A, D)$. Thus if the set has (C, D) as a pair of triprincipal points, $M_r(A, C)$ and $M_r(B, D)$ can have no common point.

Next $M_r(A, B)$ must have only one component $M_r(C, D)$. For, if it had two or more of them, then since

$$M_{s_1}(A, B) + M_r(A, C) + M_r(B, D)$$

and

$$M_{s_2}(A, B) + M_r(A, C) + M_r(B, D)$$

are components having (C, D) as a pair of principal points, and different from each other and also from $M_r(C, D)$, the set would have four or more components having (C, D) as a pair of principal points, so that (C, D) can not be a pair of triprincipal points.

Cor. In a simple continuous set with respect to a pair of triprincipal points (A, B) , the necessary and sufficient condition that two interior points C, D of one constituent component $M_r(A, B)$ should be a pair of triprincipal points is that $M_r(A, C)$ and $M_r(B, D)$ should have no common point, where C is an interior point of $M_r(A, D)$.

Remark. Two conditions stated in Theorem 199 are independent of each other. For, there is a set $M(A, B)$ whose components $M(A, C)$ and $M(B, D)$ have no common point, while it has two components $M_1(C, D)$ and $M_2(C, D)$. For example, take a set defined by the equations

$$\text{i) } \rho = \frac{\theta}{1 + \theta} \quad (0 \leq \theta)$$

$$\text{ii) } \rho = 1 \quad (0 \leq \theta \leq 2\pi)$$

$$\text{iii) } 1 \leq \rho \leq 2 \quad (0 = \theta),$$

and the four points

$$A(\rho=2, \theta=0), \quad B(\rho=0, \theta=0), \quad C\left(\rho=\frac{3}{2}, \theta=0\right), \quad D\left(\rho=1, \theta=\frac{\pi}{4}\right)$$

of the set, then it is easily seen that the set has the said property. Moreover there is a set $M(A, B)$ which contains only one component $M(C, D)$, and yet whose components $M(A, C)$ and $M(B, D)$ have common points, where C is an interior point of $M(A, D)$. For example, take a set defined by the equations

$$\text{i) } y = \sin \frac{\pi}{1-x} \quad (0 \leq x < 1)$$

$$\text{ii) } y = (+1, -1) \quad (x=1)$$

$$\text{iii) } y=0 \quad (1 < x \leq 2)$$

and the four points

$$A(x=0, y=0), \quad B(x=2, y=0), \quad C(x=1, y=1), \quad D(x=1, y=-1)$$

of the set, then the set has the said property.

Theorem 200. If, in a continuous set with a pair of triprincipal points, any two points of constituent component be a pair of triprincipal points, then the component is a Jordan curve.

Proof. By the previous theorem, in order that any two points C, D of a constituent component $M_r(A, B)$ may be a pair of triprincipal points, it is necessary that $M_r(A, B)$ has only one component having (C, D) as a pair of principal points, namely the component is simple. Further $M_r(A, C)$ and $M_r(B, D)$ can have no common point when C is an interior point of $M_r(A, D)$, so that the simple component $M_r(A, B)$ can contain no component of the third kind. Accordingly $M_r(A, B)$ is a Jordan curve (Theorem 132).

Theorem 201. There is no set with a pair of triprincipal points, such that any two points of the set form a pair of triprincipal points.

Proof. By the previous theorem, in order that any two points of a constituent component may be a pair of triprincipal points, the component must be a Jordan curve. Thus if there were a continuous set having the said property, its three constituent components would be all Jordan curves. But in such a set, if we take a point C in one constituent component $M_1(A, B)$ and another point D in the second component $M_2(A, B)$, then we have the following distinct four components, each of which has (C, D) as a pair of principal points:

$$\begin{aligned}
M_1(C, A) + M_2(A, D) &\equiv M_1(C, D), \\
M_1(C, B) + M_2(B, D) &\equiv M_2(C, D), \\
M_1(C, B) + M_3(B, A) + M_2(A, D) &\equiv M_3(C, D), \\
M_1(C, A) + M_3(A, B) + M_2(B, D) &\equiv M_4(C, D).
\end{aligned}$$

Thus (C, D) is not a pair of triprincipal points.

In a set with a pair of biprincipal points (A, B) , it has been proved that if any two points of its constituent component form a pair of biprincipal points, then the component is a Jordan curve; and also it has been proved that conversely, a set whose constituent components are all Jordan curves, having no common point other than its biprincipal points, has such a property that any two points of it form a pair of biprincipal points of the set.

In a set with a pair of triprincipal points, the former part of the above proposition is also true as was proved in Theorem 200, but the latter part is not true (Theorem 201). Namely in such a set with triprincipal points, pairs of points may be divided into two kinds, i.e.,

- (i) pairs of triprincipal points when the points belong to one constituent component;
- (ii) pairs of quadriprincipal points, when the points belong to different constituent components.

Theorem 202. In a set with a pair of triprincipal points (A, B) , any non-principal point of a constituent component is a conjugate triprincipal point of A and B at the same time. Thus the set has an infinite number of systems of three points, every two of which form a pair of triprincipal points.

Proof. Let R be a non-principal point of $M_1(A, B)$, then $M_1(A, B)$ has one and only one component $M_1(B, R)$. Next since $M_1(R, A)$ is a proper component of $M_1(A, B)$ and so does not contain B , $M_1(R, A)$ and $M_2(A, B)$ have only one common point A , and therefore the sum of them has (B, R) as a pair of principal points; denote this sum by $M_2(B, R)$. Similarly the sum of $M_1(R, A)$ and $M_3(A, B)$ has (B, R) as a pair of principal points and so may be denoted by $M_3(B, R)$. It is clear that there is no other component having (B, R) as a pair of principal points, and moreover the sum of the above three components is identical with the whole set. So (B, R) is a pair of triprincipal points of the set. Similarly it can be proved that (A, R) is also a pair of triprincipal points of the set.

Further from the fact that a set having a non-principal point has an infinite number of them, the latter part of the theorem follows.

Theorem 203. If, in an ordinary set with a pair of triprincipal points (A, B) , one of its constituent components be of the third kind, then any conjugate principal point of A (or B) is also a conjugate triprincipal point of A (or B), but not of both at the same time.

Proof. Denote by $M_1(A, B)$ the constituent component of the third kind, and by P a conjugate principal point of A with respect to that component; then, since $M_1(A, B)$ is ordinary, $M_1(B, P)$ does not contain A . Thus $M_2(A, B) + M_1(B, P)$ is a continuous set having (A, P) as a pair of principal points; denote it by $M_2(A, P)$. Similarly $M_3(A, B) + M_1(B, P)$ is a continuous set having the same property; denote it by $M_3(A, P)$. These two components and $M_1(A, P) \equiv M_1(A, B)$ are different from one another, and sum of them is clearly identical with the whole set. Moreover it is obvious that there is no other component having (A, P) as a pair of principal points. Thus P is a conjugate triprincipal point of A .

On the contrary, the set has only one component having (B, P) as a pair of principal points, namely that belonging to $M_1(A, B)$. For, P being a point belonging to $M_1(A, B)$ only, any component $M(B, P)$ must contain $M_1(B, P)$ or $M_1(A, P) \equiv M_1(A, B)$ and accordingly always $M_1(B, P)$. Thus P cannot be a conjugate triprincipal point of B . Q.E.D.

From Theorems 202 and 203, we have the following theorem.

Theorem 204. In an ordinary set with a pair of triprincipal points (A, B) , any point of it is a conjugate triprincipal point of at least one of A, B .

Definition 18. By Theorem 12, any point P of a singular set $M(A, B)$ is a conjugate principal point of one of A, B , or of both. In the former case, the point P is called a semi-principal point of the singular set $M(A, B)$, and in the latter case a perfect principal point of the singular set. If we apply this definition to an ordinary set, then any principal point of the set is a semi-principal one.

Theorem 205. If in a set with a pair of triprincipal points (A, B) , its constituent component $M_r(A, B)$ be singular, then any semi-principal point of it is a conjugate triprincipal point of one and only one of A, B ; but any perfect principal point of it, none of A, B .

Proof. This theorem may be proved in exactly the same manner as in Theorem 204.

From Theorems 203, 204 and 205, we have the following important relations in a continuous set with a pair of triprincipal points (A, B) .

(i) If P be a non-principal point of constituent components, then it is a conjugate triprincipal point of A and B at the same time.

(ii) If P be a *semi-principal point* of constituent components, then it is conjugate triprincipal point of one and only one of A, B .

(iii) If P be a *perfect principal point* of constituent components, then it is a conjugate triprincipal point of none of A, B .

Further we may prove the following theorem.

Theorem 206. When P is a *perfect principal point* of a constituent component, it cannot be a conjugate triprincipal point of any other point of the set; in other words, P is a *non-triprincipal point* of the set.

Proof. Since P is a *perfect principal point* of a constituent component $M_1(A, B)$, the component $M_1(A, B)$ must be singular, having (A, B, P) as a singular system of three points. Therefore the set cannot contain more than two components having (P, Q) as a pair of principal points, where Q denotes any point of the set. So the point P cannot be a conjugate triprincipal point of any point of the set. *Q.D.E.*

If we extend Definition 18 to the set with a pair of triprincipal points (A, B) , namely if we call a point P , which is a conjugate triprincipal point of both A and B , a *perfect triprincipal point*; and that which is a conjugate triprincipal point of one and only one of A, B , a *semi-triprincipal point*, then we may state the result obtained above as follows.

Theorem. In a set with a pair of triprincipal points, (i) if P be a *perfect principal point* of a constituent component, then it is a *non-triprincipal point* of the set; (ii) if P be a *non-principal point* of a constituent component, then it is a *perfect triprincipal point* of the set; (iii) if P be a *semi-principal point* of a constituent component, then it is also a *semi-triprincipal point* of the set.

From Theorems 202, 203 and 205, we have the following theorem.

Theorem. If a continuous set has a pair of triprincipal points, (its constituent components may be singular or ordinary, simple or non-simple), then it has an infinite number of them.

Theorem 207. In a set with a pair of triprincipal points (A, B) , if two points C, D be interior points of different constituent components, the necessary and sufficient condition that C, D should be a pair of triprincipal points of the set is that one and only one of C, D should be a conjugate principal point of one and only one of A, B ; or one of C, D is a conjugate principal point of one of A, B , and the other that of the other with respect to that constituent component to which C or D belongs.

Proof. To discuss the nature of a pair of points (C, D) , we shall distinguish the following cases.

- (i) The points C and D are non-principal points of $M_1(A, B)$ and $M_2(A, B)$ respectively.

In this case the four components

- (a) $M_1(A, C) + M_3(A, B) + M_2(B, D)$,
- (b) $M_1(B, C) + M_3(A, B) + M_2(A, D)$,
- (c) $M_1(A, C) + M_2(A, D)$,
- (d) $M_1(B, C) + M_2(B, D)$

are distinct continuous ones having (C, D) as a pair of principal points, and sum of them is the original set itself. Moreover it is clear that there is no other component having (C, D) as a pair of principal points, so (C, D) is a pair of quadriprincipal points of the set.

- (ii) One of C, D is a semi-principal point and the other a non-principal point of the constituent components.

Suppose that C is a conjugate principal point of A , then $M_1(A, C)$ is identical with $M_1(A, B)$, and accordingly (a) contains (d). Thus there are only three components having (C, D) as a pair of principal points, and sum of them is identical with the original set itself. So in this case, (C, D) is a pair of triprincipal points of the set.

- (iii) The points C and D are semi-principal points of $M_1(A, B)$ and $M_2(A, B)$ respectively.

This case may be again subdivided into two cases.

First case. C and D are conjugate principal points of A and B (or B and A) respectively.

In this case

$$\begin{aligned} M_1(A, C) &\equiv M_1(A, B), \\ M_2(B, D) &\equiv M_2(A, B), \end{aligned}$$

and therefore (a) is the whole set, but (b), (c), (d) are distinct components having (C, D) as a pair of principal points, and sum of them is identical with the whole set. So in this case, (C, D) is a pair of triprincipal points of the set.

Second case. C and D are conjugate principal points of one of A, B .

Suppose that C, D are conjugate principal points of A , then (a), (b) and (c) contain (d), so in this case there are only one component having (C, D) as a pair of principal points, and of course (C, D) cannot be a pair of triprincipal points of the set.

- (iv) At least one of C, D is a perfect principal point of a constituent component.

Suppose that C is a perfect principal point of $M_1(A, B)$ with respect to A, B , then

$$M_1(A, C) \equiv M_1(A, B),$$

$$M_1(B, C) \equiv M_1(A, B).$$

In this case, (a) contains (d), and (b) contains (c); accordingly there are at most two distinct components having (C, D) as a pair of principal points, and sum of them is less than the whole set. So in this case, (C, D) is not a pair of triprincipal points.

Thus, of all possible cases, only in the case (ii) and the first of the case (iii), (C, D) is a pair of triprincipal points. Hence the validity of our theorem follows.

Theorem 208. If, in a set with a pair of triprincipal points (A, B) , a point C of $M_r(A, B)$ be a conjugate triprincipal point of A , then all points of $M_r(B, C)$ are conjugate triprincipal points of A .

Proof. Take any point P of $M_r(B, C)$, then (B, P) cannot be a pair of principal point of $M_r(A, B)$. For, if so, then

$$M_r(B, P) \equiv M_r(B, A),$$

and also since $M_r(B, C)$ contains $M_r(B, P)$, we have

$$M_r(B, C) \equiv M_r(B, P) \equiv M_r(B, A).$$

Therefore $M_r(A, C)$ is a set of conjugate principal points of B with respect to $M_r(A, B)$, or is identical with $M_r(A, B)$, and thus the set has only one component $M_r(A, C)$ having (A, C) as a pair of principal points, which contradicts the hypothesis that (A, C) is a pair of triprincipal points. Consequently P is either non or semi-principal point of $M_r(A, B)$. So, by Theorems 202, 203 and 205, P is a conjugate triprincipal point of A .

This theorem is true for a singular set as well as an ordinary one.

Case II.

Now we proceed to consider the case, in which constituent components of a set have common points other than its triprincipal points, and to study its fundamental properties, which may be extended to the general theory of continuous set with a pair of n -ple principal points in the next Part.

The set discussed here is an ordinary simple continuous set with respect to a pair of triprincipal points, unless the contrary is stated.

Theorem 209. If, in a set with a pair of triprincipal points (A, B) , its constituent components have a common point C , which is a conjugate principal point of A with respect to all constituent components, then (A, C) is a pair of triprincipal points of the set. In this case, the three constituent components may have an infinite number of common points.

Proof. By hypothesis, we have

$$M_1(A, C) \equiv M_1(A, B),$$

$$M_2(A, C) \equiv M_2(A, B),$$

and

$$M_3(A, C) \equiv M_3(A, B).$$

Now it is clear that there is no other component having (A, C) as a pair of principal points, than those mentioned above, when there is no other common point than C . So in this case, (A, C) is a pair of triprincipal points. But when there are other common points, we must proceed as follows to prove the theorem.

Suppose that there were a fourth component having (A, C) as a pair of principal points, and denote it by $M_4(A, C)$. Also denote by S_r ($r=1, 2, 3, 4$) the aggregate of conjugate principal points of A with respect to $M_r(A, C)$, then $M_1(A, C)$ does not contain certain points of $M_s(A, C) - S_s$ ($s=1, 2, 3$), since, if it contained all points of $M_s(A, C) - S_s$, it would also contain $M_s(A, C)$ itself (Theorem 39). By Theorem 36, S_s is a continuous set and contains B and C , and accordingly a continuous component $M_s(B, C)$; This component $M_s(B, C)$ having contained no point of $M_s(A, C) - S_s$, and also of $M_t(A, C) - S_t$ ($t=1, 2, 3, t \neq s$), the sum of $M_1(A, C)$ and $M_s(B, C)$ does not contain certain points of $M_t(A, C)$ ($t=1, 2, 3$); and so a component $M_4(A, B)$, contained in $M_4(A, C) + M_s(B, C)$, is different from any one of $M_1(A, C) \equiv M_1(A, B)$, $M_2(A, C) \equiv M_2(A, B)$, and $M_3(A, C) \equiv M_3(A, B)$, but this contradicts the hypothesis that (A, B) is a pair of triprincipal points of the set.

Cor. In the set of the previous theorem, (B, C) cannot be a pair of triprincipal points of the set.

Proof. Any continuous component $M_k(B, C)$ having (B, C) as a pair of principal points cannot contain A . For, if it contained it, then it would contain a component $M_k(A, B)$, but as the set has only three components $M_1(A, B)$, $M_2(A, B)$ and $M_3(A, B)$ having (A, B) as a pair of principal points, $M_k(A, B)$ must be identical with one of them, say

(1) For, if $M_t(A, C) - S_t$ had a common point P with $M_s(B, C)$, the sum of $M_t(A, P)$ and $M_s(B, P)$ would contain a component having (A, B) as a pair of principal points, denote it by $M_4(A, B)$. This component $M_4(A, B)$ is different from $M_1(A, B)$ and $M_3(A, B)$, since $M_t(A, P) + M_s(B, P)$ does not contain certain points of $M_1(A, B)$ and also those of $M_3(A, B)$. And of course it is different from the third component $M_2(A, B)$, since $M_t(A, P) + M_s(B, P)$ consists only of the elements of $M_1(A, B)$ and $M_3(A, B)$. Thus the set has the four different components having (A, B) as a pair of principal points, contrary to the hypothesis.

$M_1(A, B)$. So $M_k(B, C)$ will contain $M_1(B, C)$ and accordingly be identical with it, from which follows that

$$M_k(B, C) \equiv M_1(B, C) \equiv M_1(A, B) \equiv M_1(A, C),$$

contrary to the hypothesis that the set is ordinary. Thus the sum of any components having (B, C) as a pair of principal points cannot contain A and so cannot be identical with the given set.

Theorem 210. In a set with a pair of triprincipal points (A, B) , any common point of the three constituent components is a conjugate principal point of A (or B), but not of B (or A) with respect to all the constituent components at the same time, provided that any two of the constituent components do not contain a common continuous component containing one of A, B .

Proof. In the first place, the common point C must be a conjugate principal point of A or B with respect to $M_1(A, B)$. For, if not so, first suppose that C is also not a conjugate principal point of A with respect to $M_2(A, B)$, then since $M_2(A, C) + M_1(C, B)$ does not contain the points of $M_1(A, B)$ in the neighborhood of A , and the points of $M_2(A, B)$ in the neighborhood of B , $M(A, B)$, contained in $M_2(A, C) + M_1(C, B)$, is different from $M_1(A, B)$ and $M_2(A, B)$. So the set will have the four different components $M_1(A, B)$, $M_2(A, B)$, $M_3(A, B)$ and $M_4(A, B)$ having (A, B) as a pair of principal points, contrary to the hypothesis.

Next suppose that C is a conjugate principal point of A with respect to $M_2(A, B)$, then, since $M_2(A, B)$ is ordinary, (B, C) is not a pair of principal points of $M_2(A, B)$, and accordingly $M_2(B, C) + M_1(C, A)$ contains a component having (A, B) as a pair of principal points, which is different from $M_1(A, B)$ and $M_2(A, B)$. Thus in this case also it leads us to a contradiction as before. Consequently C must be a conjugate principal point of A or B with respect to $M_1(A, B)$. Similarly it must be so with respect to $M_2(A, B)$.

Now the point C must be a conjugate principal point of A (or B) with respect to the both components $M_1(A, B)$ and $M_2(A, B)$, since, if otherwise and it be a conjugate of A with respect to $M_1(A, B)$ and of B with respect to $M_2(A, B)$, then $M_1(B, C) + M_2(C, A)$ would contain a component $M(A, B)$, different from $M_1(A, B)$ and $M_2(A, B)$, contrary to the hypothesis.

Let C be a conjugate principal point of A with respect to the both components, then C cannot be a conjugate principal point of B with respect to the same components since the set is ordinary.

Now by the same reasoning as before, we can prove that C is a conjugate principal point of A or B with respect to $M_1(A, B)$ and $M_3(A, B)$ at the same time. But C being a conjugate of A , but not of B with respect to $M_1(A, B)$, it must be so with respect to $M_3(A, B)$. Thus C is a conjugate principal point of A with respect to the three constituent components at the same time. *Q.E.D.*

Cor. In a set with a pair of triprincipal points (A, B) , any common point of the three constituent components is a conjugate triprincipal point of one and only one of A, B , provided that any two of the constituent components have no common continuous component containing one of A, B .

Theorem 211. If three constituent components of a set with a pair of triprincipal points (A, B) contain two common points C and C' having the following properties:

- i) C and C' do not belong to a common continuous component containing one of A, B ;
- ii) C and C' form a pair of principal points of all the constituent components,

then all the common points of the three constituent components are conjugate principal points of one and only one of A, B with respect to all the constituent components.

Proof. Since C and C' form a pair of principal points of a constituent component, one of them, say C , is a conjugate principal point of A , and the other C' that of B with respect to the above component, therefore, by the same reasoning as in the previous theorem, we know that C is a conjugate principal point of A , and C' that of B with respect to the three constituent components at the same time. Hence all the points of $M_r(B, C)$ ($r=1, 2, 3$) are conjugate principal ones of A with respect to $M_r(A, B)$, and since C does not belong to a common continuous component containing B , so all common points of the three constituent components contained in the common continuous component containing B (if there be any) must be conjugate principal ones of A with respect to the three constituent components⁽¹⁾. Similarly all common points contained in the common continuous component containing A are also conjugate principal ones of B with respect to the three constituent components.

(1) If a point P of the common continuous component M_B were a non-conjugate principal point of A with respect to $M_r(A, B)$, then $M_r(B, P)$ would contain all conjugate principal points of A and so also would contain the point C . But since the component $M_r(A, B)$ is a simple set and M_B is a continuous component, $M_r(B, P)$ is contained in M_B , and so C would be contained in M_B , contrary to the hypothesis. Therefore all points of M_B are conjugate principal points of A with respect to $M_r(A, B)$.

The remaining common points of the three constituent components have also the same property by the previous theorem and Theorem 169. Thus our theorem is established.

From Theorems 209 and 211, we have the following theorem.

Theorem 212. When a set with a pair of triprincipal points (A, B) contains two points C and C' having the properties stated in Theorem 211, all common points of the three constituent components form a pair of triprincipal points of the set with A or B .

Cor. 1. In a set with a pair of triprincipal points (A, B) , when any two of its three constituent components have neither a common continuous component containing A nor that containing B , all the points which are common to the three constituent components may be divided into two aggregates, such that all elements of one aggregate are conjugate principal points of A with respect to the three constituent components at the same time, while all elements of the other are those of B with respect to the same components.

Cor. 2. In the above set, any non-principal point of the constituent components cannot be a common point of the three constituent components, nor that of any two of them.

The first part follows at once from Cor. 1, and the latter part from Theorem 160, Cor. 2.

The two aggregates stated in Cor. 1 have the following interesting property.

Theorem 213. Any element of the one aggregate forms a pair of triprincipal points of the set with any element of the other, while any two elements of the same aggregate always do not form it.

Proof. Denote the two aggregates by $\{A\}$ and $\{B\}$, and take any elements A_m, B_n from them. By the previous theorem, B_n is a conjugate triprincipal point of A , but not of B ; and A_m that of B , but not of A . Now take (A, B_n) as a new pair of triprincipal points, then, since B_n is a conjugate principal point of A with respect to the three constituent components (Theorem 212 Cor.), the new constituent components are the same as old, that is,

$$M_1(A, B_n) \equiv M_1(A, B),$$

$$M_2(A, B_n) \equiv M_2(A, B),$$

$$M_3(A, B_n) \equiv M_3(A, B).$$

Therefore the new aggregates of common points of the three constituent components are also $\{A\}$ and $\{B\}$, and since all points of $\{B\}$ are conjugate triprincipal points of A , and none of $\{A\}$ is so, all points of $\{A\}$

are conjugate triprincipal points of B_n , and none of $\{B\}$ is so by the previous theorem. Accordingly A_m is a conjugate triprincipal point of B_n while any element B_i , taken from $\{B\}$, is not so. The theorem is thus proved.

Theorem 214. In the above set, any non-principal point of a constituent component is a conjugate triprincipal point of A and B at the same time.

Proof. Take any non-principal point P of a constituent component $M_1(A, B)$, then $M_1(A, B)$ has one and only one component having (A, P) as a pair of principal points; denote it by $M_1(A, P)$.

The sum of $M_2(A, B)$ and $M_1(B, P)$ has also (A, P) as a pair of principal points. For, any continuous component containing A, P in the set $M_2(A, B) + M_1(B, P)$ must contain at least one (say B_m) of the common points of $M_2(A, B)$ and $M_1(B, P)$, and accordingly must contain $M(A, B_m)$ and $M(B_m, P)$. But by the property of these common points,

$$M(A, B_m) \equiv M_2(A, B),$$

and

$$M(B_m, P) \equiv M_1(B, P).$$

That is, any continuous component containing A, P is always identical with the whole set $M_2(A, B) + M_1(B, P)$, so (A, P) is a pair of principal points of the set; denote this component by $M_2(A, P)$.

Similarly $M_3(A, B) + M_1(B, P)$ has (A, P) as a pair of principal points; denote it by $M_3(A, P)$. Now by the property of the common points that all of them are principal points of the constituent components, it is clear that there is no other component having (A, P) as a pair of principal points. Thus (A, P) is a pair of triprincipal points of the set. The same is true of another pair (B, P) . *Q. E. D.*

In the above set, if we denote by Ψ_A, Ψ_B the two aggregates of common points of the three constituent components, and by Φ that of non-principal points of them, then from Theorems 213 and 214, we have the following important theorem.

Theorem 215. In a continuous set with a pair of triprincipal points (A, B) , any two of whose constituent components have neither a common continuous component containing A nor that containing B , the three aggregates Ψ_A, Ψ_B, Φ have such a property that any two points, taken from the different aggregates, form always a pair of triprincipal points.

Theorem 216. If, in a continuous set with a pair of triprincipal points (A, B) , whose constituent components have a common continuous com-

ponent containing A and also that containing B , every one of the constituent components has these two common components as the aggregates of all the principal points of it, then the constituent components cannot have other common points.

Proof. Denote by M_A and M_B the two common continuous components in question; then, since, by hypothesis, the three constituent components have M_A and not more than M_A as a common continuous component containing A , so at least one of the constituent components must have M_A and not more than M_A as a common continuous component containing A with any one of the other constituent components. Denote it by $M_r(A, B)$. Similarly there is at least one of the constituent components having M_B and not more than M_B as a common continuous component containing B . Denote it by $M_s(A, B)$.

i). When $M_r(A, B)$ and $M_s(A, B)$ are different from each other, by Theorem 170, Cor., they have no common point other than M_A and M_B , and accordingly the common points of the three constituent components cannot be more than those of M_A and M_B .

ii). When $M_r(A, B)$ is identical with $M_s(A, B)$, by the same Cor., $M_r(A, B)$ and $M_t(A, B)$ ($t \neq r$) have no common point other than M_A and M_B , and accordingly in this case also the same conclusion as above is obtained. *Q. E. D.*

Cor. In the above set, if any two of the constituent components have M_A and M_B and not more than them as common continuous components containing one of A, B , then any two of the constituent components have no common point other than those of M_A and M_B .

From this Cor., we may deduce the following in the same manner as in the case of the set having a pair of biprincipal points.

Theorem 217. The set defined in the above Cor. may be divided into three aggregates Ψ_A , Ψ_B , and Φ having the following properties:

i) all points of Ψ_A are conjugate principal points of any point in Ψ_B , and vice versa with respect to every one of the three constituent components;

ii) all points of Φ are non-principal points of the constituent components;

iii) Ψ_A and Ψ_B are both continuous, while Φ is connected and has Ψ_A, Ψ_B as the aggregates of its limiting points;

iv) the three aggregates Ψ_A, Ψ_B, Φ have such a property that any two points, taken from their different aggregates, always form a pair of tri-principal points of the set.

Theorem 218. If, in a set with a pair of triprincipal points (A, B) , its constituent components be of the second kind, and have a common continuous component containing A and also that containing B , then the set has only one pair of triprincipal points.

Proof. Taking any two points C, D of the set, at least one of which is different from A, B , we shall prove that C, D cannot be a pair of triprincipal points. In the case when A is different from C, D , determine a point P in a common continuous component containing A , such that $M_r(B, P)$ ($r=1, 2, 3$) contains C or D or both when $M_r(A, B)$ contains C or D or both. Such a point P surely exists, for, describe a sphere with A as centre and with a radius less than d , where d denotes the smaller one of the two distances AC and AD , and consider a common continuous component $M_3(A, P)$ wholly contained in the above sphere, then $M_r(A, P)$ contains neither C nor D , so $M_r(B, P)$ must contain them.

Therefore any continuous component having (C, D) as a pair of principal points, contained in

$$M_1(A, B) + M_2(A, B) + M_3(A, B),$$

is also contained in

$$M_1(B, P) + M_2(B, P) + M_3(B, P),$$

as $M_1(A, P), M_2(A, P)$ and $M_3(A, P)$ are identical with one another and they contain neither C nor D . Now $M_r(B, P)$ does not contain A since $M_r(A, B)$ is of the second kind. Thus the sum of any components having (C, D) as a pair of principal points does not contain A , from which follows that (C, D) cannot be a pair of triprincipal points of the given set.

In the case where A coincides with one of C, D, B must be different from C, D . Thus, taking B instead of A , we arrive at the same conclusion by proceeding in a similar manner to the above. *Q. E. D.*

Theorem 219. If, in a set with a pair of triprincipal points (A, B) , its constituent components be of the second kind, having a common continuous component containing A , but no other common points except B , then the set has an infinite number of pairs of triprincipal points, one element of every pair always being A .

Proof. First, that there cannot be a pair of triprincipal points, which has not A as one of its elements, is proved in a similar manner to the previous theorem. Next, that there is an infinite number of pairs of triprincipal points is proved as follows.

In $M_1(A, B)$, take a point P , such that $M_1(B, P)$ contains no point of the common continuous component; then (A, P) is a pair of triprincipal points. For, since $M_2(A, B)$ and $M_1(B, P)$ have common point other than B , its sum has (A, P) as a pair of principal points. The same is true of the sum of $M_3(A, B)$ and $M_1(B, P)$. Thus there are three components having (A, P) as a pair of principal points, namely the previous two and the one contained in $M_1(A, B)$. It is clear that the sum of these components is identical with the whole set, and also that there is no other component having (A, P) as a pair of principal points. So (A, P) is a pair of triprincipal points of the set. It is also clear that there is an infinite number of such points as P .

This theorem may be extended as follows.

If, in a set with a pair of triprincipal points (A, B) , its constituent components be of the second kind, having a common continuous component containing A , but not that containing B , then the theorem still holds. This may be proved by using the property of a set proved in Theorem 148.

Theorem 220. If, in a set with a pair of triprincipal points (A, B) , its constituent components have a common continuous component containing B , and at least one of them is of the third kind having A as its compound principal point, then this common continuous component contains a continuous part, every point of which is a conjugate triprincipal point of A .

Proof. Let $M_1(A, B)$ be of the third kind, then it has a continuous component consisting of conjugate principal points of A in the common component containing B . Taking any one point P of it, we shall prove that P is a conjugate triprincipal point of A .

Now the three components $M_1(A, P) \equiv M_1(A, B)$, $M_2(A, P)$ and $M_3(A, P)$ are different from one another, since

$$M_1(A, P) + M_1(P, B) \equiv M_1(A, B),$$

$$M_2(A, P) + M_2(P, B) \equiv M_2(A, B),$$

$$M_3(A, P) + M_3(P, B) \equiv M_3(A, B),$$

and

$$M_1(P, B) \equiv M_2(P, B) \equiv M_3(P, B).$$

Moreover the sum of these three components is clearly identical with the whole set. Thus to prove the theorem we have only to prove that there is no other component having (A, P) as a pair of principal points.

Assume that there were a fourth component $M_4(A, P)$ having (A, P) as a pair of principal points. The sum of $M_4(A, P)$ and $M_1(P, B)$ has a component having (A, B) as a pair of principal points, denote it by

$M(A, B)$. Since the set has only three components $M_1(A, B)$, $M_2(A, B)$ and $M_3(A, B)$ having (A, B) as a pair of principal points, $M(A, B)$ must be identical with one of them.

First suppose that $M(A, B)$ is identical with $M_1(A, B)$; then $M_4(A, P) + M_1(P, B)$ contains $M_1(A, P) + M_1(P, B)$, and accordingly $M_4(A, P)$ contains at least $M_1(A, P) - \{C_1\}$, where $\{C_1\}$ denotes the aggregate of common points of $M_1(A, P)$ and $M_1(P, B)$. But since $M_1(A, P)$ is of the third kind and all points of $M_1(P, B)$ are its conjugate principal points of A , $M_4(A, P)$ must contain $M_1(A, P)$, contrary to the hypothesis.

Next suppose that $M(A, B)$ is identical with $M_2(A, B)$, then by the same reasoning as above, $M_4(A, P)$ must contain at least $M_2(A, P) - \{C_2\}$. But since $\{C_2\}$ is a continuous set (Theorem 89), $M_2(A, P) - \{C_2\}$ is not closed and so has its limiting points in $\{C_2\}$. By Theorem 129, the sum of $M_2(A, P) - \{C_2\}$ and its limiting points is a continuous set having (A, C_{2l}) as a pair of principal points, where C_{2l} is one of limiting points contained in $\{C_2\}$. Thus $M_4(A, P)$ contains at least $M_2(A, C_{2l})$ and $M_4(C_{2l}, P)$. But C_{2l}, P being points of $M_2(P, B) \equiv M_1(P, B) \equiv M_3(P, B)$, and the constituent components being simple sets, $M_4(C_{2l}, P)$ must be a component of $M_2(P, B)$, so we have

$$M_4(C_{2l}, P) \equiv M_2(C_{2l}, P).$$

That is, $M_4(A, P)$ contains $M_2(A, C_{2l}) + M_2(C_{2l}, P)$ and accordingly $M_2(A, P)$, contrary to the hypothesis.

Similarly the supposition that $M(A, B)$ is identical with $M_3(A, B)$ leads to a contradiction. Thus there cannot be a fourth component $M_4(A, P)$.

Cor. If, in a set with a pair of triprincipal points (A, B) , its constituent components have a common continuous component containing A and also that containing B , and at least one of them be of the third kind, then the set has a continuous component whose points are all conjugate triprincipal points of A or B .

Theorem 221. (i) If, in a set with a pair of triprincipal points (A, B) , whose constituent components are all Jordan curves, the three constituent components have a third common point C , then they have a common continuous component containing C and one of A, B .

(ii) The constituent components of the above set cannot have common points other than those belonging to a common continuous component containing one of A, B .

The validity of this theorem follows at once from Theorem 146 and the property of the Jordan curve.

Part. III.

Continuous Set with Principal Points of the n th Order.

Definition 19. When a continuous set has two distinct points A, B , such that there are n and only n components having A, B as a pair of principal points, and the sum of them is identical with the set itself, these points A, B are called a pair of principal points of the n^{th} order or a pair of n -ple principal points of the set.

Theorem 222. A continuous set with a pair of n -ple principal points ($n \geq 3$) cannot have a pair of uniprincipal points.

The proof may be established in exactly the same manner as in the set with a pair of triprincipal points.

Here we shall discuss the properties of an ordinary simple continuous set with respect to a pair of n -ple principal points; so under a continuous set an ordinary simple one is always understood unless specially is stated otherwise. But many of them are common to all continuous sets (ordinary or singular, simple or non-simple) having a pair of n -ple principal points.

Case I.

First we consider the case in which the constituent components have no common point other than the n -ple principal points.

Theorem 223. A continuous set with a pair of n -ple principal points cannot have a pair of principal points of the lower order ($n \geq 3$).

Proof. This theorem is proved by mathematical induction. Assuming that the theorem is true for a continuous set with $(n-1)$ -ple principal points, we shall prove that it is also true for a continuous set with n -ple principal points $M^{(n)}(A, B) \equiv M_1(A, B) + M_2(A, B) + \dots + M_{n-1}(A, B) + M_n(A, B)$.

First take any two points C, D on $M_r(A, B)$ ($r=1, 2, \dots, n$), then there may occur two cases, i) (C, D) is not a pair of principal points of any order whatever with respect to $M^{(n-1)}(A, B)$, or ii) if it be a pair of principal points of certain order, its order is at least $(n-1)^{(1)}$. In the former case, (C, D) is not also a pair of principal points of any

(¹) By assumption.

order whatever with respect to $M^{(n)}(A, B)$, since the sum of the components having (C, D) as a pair of principal points in $M^{(n-1)}(A, B)$ does not contain certain points of $M^{(n-1)}(A, B)$, and so also the same must be true for $M^{(n)}(A, B)$. In the latter case, $M_r(A, C)$ and $M_r(B, D)$, where C is an interior point of $M_r(A, D)$, cannot have a common point. Therefore $M_r(A, C) + M_p(A, B) + M_r(B, D)$, where $M_p(A, B)$ denotes the component not contained in $M^{(n-1)}(A, B)$, is a component having (C, D) as a pair of principal points. Now $M^{(n-1)}(A, B)$ having at least $(n-1)$ components having (C, D) as a pair of principal points, so $M^{(n)}(A, B)$ has at least n of them, and the sum of them is identical with the set itself. Thus (C, D) cannot be a pair of principal points of an order lower than n .

Next take a point C on $M_r(A, B)$ and D on $M_s(A, B)$. If (C, D) be not a pair of principal points of any order whatever with respect to $M^{(n-1)}(A, B)$, then the same is true for $M^{(n)}(A, B)$. If (C, D) be a pair of principal points of certain order with respect to $M^{(n-1)}(A, B)$, then at least one of the pairs of components $[M_r(A, C), M_s(B, D)]$, $[M_r(B, C), M_s(A, D)]$ has no common points between its elements. For, if they have, it must be A or B , and therefore

$$\text{i) } M_r(A, C) \equiv M_r(A, B), \text{ or } M_s(B, D) \equiv M_s(A, B)$$

$$\text{ii) } M_r(B, C) \equiv M_r(A, B), \text{ or } M_s(A, D) \equiv M_s(A, B).$$

In the case i) $M_r(A, C) + M_s(B, D)$ has a component having (C, D) as a pair of principal points, denote it by $M'(C, D)$; then

$$M_r(A, C) + M_t(A, B) + M_s(B, D)$$

contains always the same component $M'(C, D)$ for any value of t . Similarly from the case ii) it results that

$$M_r(B, C) + M_t(A, B) + M_s(A, D)$$

contains always the same component $M''(C, D)$ for any value of t . Thus (C, D) cannot be a pair of principal points of any order whatever for the set $M^{(n-1)}(A, B)$, contrary to the hypothesis. Therefore at least one of the above two pairs has no common point between its elements. Now suppose that the one which has no common point to be $[M_r(A, C), M_s(B, D)]$, then

$$M_r(A, C) + M_t(A, B) + M_s(B, D)$$

has (C, D) as a pair of principal points, and hence $M^{(n)}(A, B)$ has at least n components having (C, D) as a pair of principal points, whose sum is identical with the whole set.

Thus the theorem is true for the set with n -ple principal points, provided that the theorem is true for the set with $(n-1)$ -ple principal points. But it was proved that the theorem is true for the set with triprincipal points, as it is true for the set with n -ple principal points, where n denotes any integer greater than 3.

Theorem 224. A set with a pair of n -ple principal points cannot have a pair of principal points of an order higher than $2(n-1)$.

Proof. It is clear that, when any two interior points are taken in a constituent component, they cannot be a pair of principal points of an order higher than n .

Further take a point C in $M_r(A, B)$, and another point D in $M_s(A, B)$, then since $M_r(A, B)$ and $M_s(A, B)$ have only two common points A, B , $M_r(A, B) + M_s(A, B)$ cannot have more than two components having (C, D) as a pair of principal points. Also there cannot be more than two components having (C, D) as a pair of principal points, which contain the points of $M_t(A, B)$, namely

$$\begin{aligned} M_r(C, A) + M_t(A, B) + M_s(B, D) &\equiv M_{t,1}(C, D), \\ M_r(C, B) + M_t(B, A) + M_s(A, D) &\equiv M_{t,2}(C, D). \end{aligned}$$

Thus, on the whole, there cannot be more than $2(n-1)$ components having the said property.

The following nine theorems may be proved in a similar manner to the case of the set with triprincipal points.

Theorem 225. In a set (ordinary or singular) with a pair of n -ple principal points (A, B) , the necessary and sufficient condition that any two interior points C, D of a constituent component $M_r(A, B)$ should be a pair of n -ple principal points of the set is that $M_r(A, C)$ and $M_r(B, D)$ should have no common point, where C is an interior point of $M_r(A, D)$.

Theorem 226. If, in a set (ordinary or singular) with a pair of n -ple principal points (A, B) , two points C, D be interior points of different constituent components, the necessary and sufficient condition that C, D should be a pair of n -ple principal points of the set is that one and only one of C, D should be a conjugate principal point of one and only one of A, B ; or one of C, D should be a conjugate principal point of one of A, B and the other that of the other.

Theorem 227. In a set with a pair of n -ple principal points, if any two points of its constituent component form always a pair of n -ple principal points, then the component is a Jordan curve.

Theorem 228. There is no set with a pair of n -ple principal points,

($n > 2$), such that any two points of it form a pair of n -ple principal points.

Theorem 229. A set (ordinary or singular) with a pair of n -ple principal points has an infinite number of pairs of them.

Theorem 230. In a set with a pair of n -ple principal points (A, B) , any non-principal point of a constituent component is a conjugate n -ple principal point of A and B at the same time.

Theorem 231. If, in a set with a pair of n -ple principal points (A, B) , one of its constituent components be of the third kind, then any conjugate principal point of A (or B) is also a conjugate n -ple principal point of A (or B), but not of both at the same time.

Theorem 232. If, in a set with a pair of n -ple principal points (A, B) , one of its constituent components be singular, then any semi-principal point of it is a conjugate n -ple principal point of one and only one of A, B ; and any perfect principal point, none of A, B .

Theorem 233. If, in a set (ordinary or singular) with a pair of n -ple principal points (A, B) , a point C of $M_r(A, B)$ is a conjugate n -ple principal point of A , then all points of $M_r(B, C)$ are the conjugate n -ple principal points of A .

Case II.

Now we proceed to consider the case, in which the constituent components have common points other than its n -ple principal points. The following ten theorems concerning this case may be proved in a similar manner to the case of the set with triprincipal points.

Theorem 234. If, in a set with a pair of n -ple principal points (A, B) , its constituent components have a common point C , which is a conjugate principal point of A with respect to all the constituent components, then (A, C) is a pair of n -ple principal points of the set.

Theorem 235. In a set with a pair of n -ple principal points (A, B) , any common point of the n constituent components is a conjugate principal point of A (or B), but not of B (or A), with respect to all the constituent components at the same time, provided that any two of the constituent components do not contain a common continuous components containing one of A, B .

Theorem 236. If all constituent components of a set with a pair of n -ple principal points (A, B) contain two common points C and C' having the properties:

- i) C and C' form a pair of principal points of all constituent components;

- ii) C and C' do not belong to a common continuous component containing one of A, B ;

then all common points are conjugate principal points of one and only one of A, B with respect to all the constituent components.

Theorem 237. When a set with a pair of n -ple principal points contains the two points C and C' having the properties stated in the above theorem, all common points of all constituent components form a pair of n -ple principal points of the set with A or B .

Cor. 1. In a continuous set with a pair of n -ple principal points (A, B) , when any two of its constituent components have neither a common continuous component containing A nor that containing B , all points which are common to n constituent components, are divided into two aggregates, such that all elements of one aggregate are conjugate principal points of any one element of the other with respect to every one of n constituent components, while any two elements of the same aggregate do not form a pair of principal points of any constituent component.

Cor. 2. In the above set, any non-principal point of the constituent components cannot be a common point of them.

In the above set, if we denote by Ψ_A, Ψ_B , the two aggregates of common points of the n constituent components, and by Φ that of non-principle points of them, we have the following theorem.

Theorem 238. In a continuous set with a pair of n -ple principal points (A, B) , any two of whose constituent components have neither a common continuous component containing A nor that containing B , the three aggregates Ψ_A, Ψ_B, Φ have such a property that any two points, taken from different aggregates, form always a pair of n -ple principal points.

Theorem 239. A continuous set with a pair of n -ple principal points (A, B) , whose constituent components have the same aggregate of principal points, and no more common continuous components containing one of A, B , may be divided into three parts Ψ_A, Ψ_B and Φ having the following properties:

- i) all points of Ψ_A are conjugate principal points of any point in Ψ_B and vice versa with respect to every one of the n constituent components;
- ii) any point of Φ is a non-principal point of the constituent components;
- iii) Ψ_A and Ψ_B are both continuous, while Φ is connected and has Ψ_A and Ψ_B as the aggregates of its limiting points;
- iv) the three aggregates have such a property that any two points,

taken from their different aggregates, always form a pair of n -ple principal points of the set.

Theorem 240. If, in a set with a pair of n -ple principal points (A, B) , its n constituent components be of the second kind, and have a common continuous component containing A and also that containing B , then the set has only one pair of n -ple principal points.

Theorem 241. If, in a set with a pair of n -ple principal points (A, B) , its n constituent components are all of the second kind, having a common continuous component containing A , but no other common point except B , then the set has an infinite number of pairs of n -ple principal points, one element of every pair being always A .

Theorem 242. If, in a set with a pair of n -ple principal points (A, B) , its n constituent components have a common continuous component containing B , and at least one of them is of the third kind having A as its compound principal point, then this common continuous component has a continuous part, every point of which is a conjugate n -ple principal point of A .

Theorem 243. i) If, in a set with a pair of n -ple principal points (A, B) , whose constituent components are all Jordan curves, the n constituent components have a third common point C , then they have a common continuous component containing C and one of A, B .

ii) The constituent components of the above set cannot have common points other than those belonging to a common continuous component containing one of A, B .

Principal Points of an Infinite Order.

We have discussed the property of a continuous set with principal points of a finite order, but we may consider a set with principal points of an infinite order.

Describe a semi-circle AB with unit radius, and join the centre O and the middle point C of the arc AB . On the straight line OC , determine a set of points $\{C_n\}$, such that

$$OC_n = 1 - \frac{1}{2^n} \quad (n=1, 2, 3, \dots),$$

and describe a circular arc through AC_nB . Then the set of these circular arcs is clearly a continuous set of points, and the number of components having (A, B) as a pair of principal points is countably infinite, and the sum of them is the set itself. This set is an example of a continuous set with a pair of principal points of a countably infinite order.

Similarly we can construct a continuous set with a pair of principal points of a non-countably infinite order. For example, construct a perfect set of points dense nowhere on the line OC , and through these points and A, B , draw circular arcs as before. Then the set of all these circular arcs is the required one, having (A, B) as a pair of principal points of a non-countably infinite order.

From this point of view, many ordinary surfaces and solids may be considered as a continuous set of points having a pair of principal points of a non-countably infinite order. But there are also surfaces and solids (somewhat different from the ordinary ones) having no pair of points of any order whatever. For example, the set of points defined by the equations,

$$\begin{aligned} \text{i)} \quad & -1 \leq y \leq +1, & -1 \leq x \leq +1, \\ \text{ii)} \quad & +1 \leq y \leq +2, & x = 0, \\ \text{iii)} \quad & y = \frac{3}{2}, & 0 \leq x \leq +1. \end{aligned}$$

Classification of Continuous Sets.

We have already classified the continuous sets into three kinds according to the number of pairs of principal points. The sets of the first kind, which have no pair of principal points, may again be subdivided into two classes according as they have or have not a pair of principal points of a certain order.

When a set has a pair of principal points of the n^{th} order, and its constituent components have no common point other than the pair of principal points, the set has no pair of principal points of an order lower than n ($n > 2$) (Theorem 223). But when the constituent components have other common points than the pair of principal points, this is never the case as may easily be seen from Theorem 224⁽¹⁾. Indeed many sets may have pairs of principal points of different orders.

Definition 20. If the lowest order of the pairs of principal points in such a set be m , we shall call this set a continuous set of the m^{th} order, and a set having no pair of principal points of any order whatever that of zero order. The continuous set of zero order may be again subdivided by the following characteristics.

(¹) When n constituent components of the set $M^{(n)}(A, B)$ stated in Theorem 224 are Jordan curves, then (C, D) is a pair of principal points of the $2(n-1)^{\text{th}}$ order. Thus if we consider this set as one having (C, D) as a pair of $2(n-1)$ -ple principal points, then this set has a pair of principal points (A, B) whose order is lower than $2(n-1)$.

Theorem 244. Among the ordinary continuous set having no pair of principal points of any order whatever, there is such a set, that it contains proper components having pairs of principal points of the first, the second, the third,, the $2n^{\text{th}}$ order, but not that of the $(2n+1)^{\text{th}}$ order.

Proof. Consider $2n$ Jordan curves having (A, B) as a pair of principal points, and denote them by $M_1(A, B)$, $M_2(A, B)$,, $M_{2n}(A, B)$. Suppose that these sets have two common continuous components having A, B as one of its principal points, namely that the sets have two common points C, D such that

$$\begin{aligned} M_1(A, C) &\equiv M_2(A, C) \equiv \dots \equiv M_{2n}(A, C), \\ M_1(B, D) &\equiv M_2(B, D) \equiv \dots \equiv M_{2n}(B, D); \end{aligned}$$

and also suppose that they have no other common point. Then clearly the set of points consisting of the sum of these $2n$ curves has (A, B) as a pair of its principal points of $2n^{\text{th}}$ order. This set contains proper components having a pair of principal points of any order less than $2n$, since $M_1(A, B) + M_2(A, B) + \dots + M_{2n-r}(A, B)$ is a component having (A, B) as a pair of principal points of $(2n-r)^{\text{th}}$ order.

Now taking a point P (other than A) in $M_1(A, C)$, construct a Jordan curve $M(P, Q)$, which has no common point other than P with the set above mentioned, then the sum of $M(P, Q)$, $M_1(A, B)$,, $M_{2n}(A, B)$ is the set having the required property.

In the first place, this set has no pair of principal points of any order whatever. For, take a point E of $M(P, Q)$, and any other point F of the set except Q , then any component having E, F as a pair of principal points cannot contain Q , so the sum of these components is not identical with the whole set, and therefore (E, F) is not a pair of principal points of any order whatever. Thus it follows that one element of principal points of a certain order (if it exist) must be Q . Now take any point G of $M_r(A, P)$, then there is only one component having (G, Q) as a pair of principal points, and clearly it does not contain certain points of the set, so that (G, Q) is not a pair of principal points of the set. Next if we take any point H other than those of $M_r(A, P)$, then any component having (H, Q) as a pair of principal points does not contain $M_r(A, P) - P$. Thus there cannot be a pair of principal points of any order whatever.

Secondly, the set has proper components having pairs of principal points of the first, the second,, the $2n^{\text{th}}$ order as was already seen.

Lastly, the set cannot contain a proper component having a pair of principal points of the $(2n+1)^{\text{th}}$ order. If, in the proper component $M_1(C, D) + M_2(C, D) + \dots + M_{2n}(C, D)$, we take E on $M_r(C, D)$ and F on $M_s(C, D)$, then (E, F) is a pair of principal points of the $2(2n-1)^{\text{th}}$ order, and, omitting one constituent component or its proper part, we obtain a proper component having a pair of principal points of the $2(2n-1)^{\text{th}}$ order diminished by 2. Thus there is no proper component having a pair of principal points of an odd degree. When we omit one or two of $M_r(C, E)$, $M_r(D, E)$, $M_s(C, F)$ and $M_s(D, F)$, or its proper part, we obtain a component having a pair of principal points of an order lower than $2n$. If we take E, F on one component $M_r(C, D)$, then (E, F) is a pair of principal points of $2n^{\text{th}}$ order. Similarly it may easily be seen that any other proper component cannot have a pair of principal points of the $(2n+1)^{\text{th}}$ order.

Theorem 245. Among the singular continuous sets having no pair of principal points of any order whatever, there is such a set, that it contains proper components having pairs of principal points of the first, the second, , the n^{th} order, but not that of the $(n+1)^{\text{th}}$ order.

Proof. Consider a continuous set of points, consisting of $M_1(A, B)$, $M_2(A, B)$, , $M_n(A, B)$, each of which is singular and has no common point other than A, B with any other, then this set has a pair of n -ple principal points (A, B) ; and omitting $M_1(A, B)$, $M_2(A, B)$, , $M_r(A, B)$, we get a proper component having (A, B) as a pair of principal points of $(n-r)^{\text{th}}$ order.

We shall prove that this set cannot have a pair of principal points of order higher than n . Take first any two points C, D on $M_r(A, B)$, then $M_1(A, B) + M_r(A, B)$ has at most two components having (C, D) as a pair of principal points, namely one in $M_r(A, B)$, and, if possible, one in $M_r(A, C) + M_1(A, B) + M_r(B, D)$ or $M_r(A, D) + M_1(A, B) + M_r(B, C)$. Thus there cannot be more than n components having (C, D) as a pair of principal points.

Next take C on $M_r(A, B)$ and D on $M_s(A, B)$, then $M_r(A, B) + M_s(A, B)$ has at most two components having (C, D) as a pair of principal points. Further in the sum of $M_r(A, B)$, $M_s(A, B)$ and $M_t(A, B)$, consider two components

- (i) $M_r(A, C) + M_t(A, B) + M_s(B, D)$,
- (ii) $M_r(B, C) + M_t(A, B) + M_s(A, D)$.

If at least one pair of (A, C) and (B, D) be that of principal points

with respect to $M_r(A, B)$ and $M_s(A, B)$, then (i) contains $M_r(B, C) + M_s(B, D)$ or $M_r(A, C) + M_s(A, D)$, and accordingly the component having (C, D) as a pair of principal points, contained in (i), will be the same as that contained in $M_r(A, B) + M_s(A, B)$. If both pairs (A, C) and (B, D) be not the pairs of principal points, then (i) has a new component having (C, D) as a pair of principal points, but, in this case, (B, C) and (A, D) being pairs of principal points with respect to $M_r(A, B)$ and $M_s(A, B)$ respectively, a component contained in (ii), which has (C, D) as a pair of principal points, will be the same as that contained in $M_r(A, B) + M_s(A, B)$. So (i) and (ii) contain at most one component having the said property, except those contained in $M_r(A, B) + M_s(A, B)$. Accordingly the set has at most n of these components, and therefore the order of principal points is at most n .

Now consider two Jordan curves $M(A, P)$ and $M(Q, R)$, such that they have only one common point R , and that $M(A, P) + M(Q, R)$ and $M_1(A, B) + M_2(A, B) + \dots + M_n(A, B)$ have only one common point A . Then the sum of $M(A, P)$, $M(Q, R)$, $M_1(A, B)$, $M_2(A, B)$, \dots , $M_n(A, B)$ is a continuous set having the property stated in the theorem.

Definition 21. If a continuous set having no pair of principal points of any order whatever contains proper components having pair of principal points of every order less than or equal to n , but never that of $(n+1)^{\text{th}}$ order, then the set is called a continuous one of the n^{th} degree and zero order.

Thus continuous sets having no pair of principal points of any order whatever may be classified by the degree of them. If we admit the continuous set of an infinite degree, then it is clear that any continuous set of zero order belongs to a set of a certain degree.

Hence we have the following table of classification of continuous sets of points.

Table of Classification.

(I)

Continuous Sets of Points.	Set of the first kind. (having no pair of principal points.)	The first class (having no pair of principal points of any order whatever.)	The first degree. The second degree. The n^{th} degree. (n may be infinite.)
			The second order. The third order. The n^{th} order. (n may be infinite.)
	Set of the second kind. (having only one pair of principal points.)	The second class (having a pair of principal points of a certain order.) (having no compound principal point.)	The first species. (Singular set.) (having no non-principal point.)
			The second species. (having an infinite number of non-principal points.)
	Set of the third kind. (having an infinite number of pairs of principal points.)	The first class. (having only one compound principal point.) The second class. (having an infinite number of compound principal points.)	

N.B.—The sets of the second and the third kinds belong to the set of the first order.

Or from another point of view, namely from the singularity or non-singularity of the sets, they may be classified into three kinds as follows.

(II)

$$\text{Continuous sets} \left\{ \begin{array}{l} \text{Ordinary set.} \\ \text{Semi-singular set.} \\ \text{Singular set.} \end{array} \right.$$

Classification of Points of a Continuous Set.

The points of a continuous set may be classified into the three kinds from the two points of view, as was already seen.

$$(I) \left\{ \begin{array}{l} \text{Non-principal point.} \\ \text{Simple principal point.} \\ \text{Compound principal point.} \end{array} \right.$$

And

$$(II) \left\{ \begin{array}{l} \text{Non-separating point.} \\ \text{Perfectly separating point.} \\ \text{Imperfectly separating point..} \end{array} \right. \left\{ \begin{array}{l} \text{The first kind.} \\ \text{The second kind.} \end{array} \right.$$

And their relation may be expressed as follows :

$$\text{Points of continuous set} \left\{ \begin{array}{l} \text{Simple principal point.} \\ \text{Compound principal point.} \\ \text{Non-principal point} \end{array} \right. \left\{ \begin{array}{l} \dots \text{Non separating point.} \\ \text{Perfectly separating point.} \\ \text{Imperfectly separating point.} \end{array} \right.$$

When a continuous set of points has a pair of principal points, the points of the set may be classified into another three kinds with respect to that pair.

$$(III) \left\{ \begin{array}{l} \text{Perfect principal point.} \\ \text{Semi-principal point.} \\ \text{Non-principal points.} \end{array} \right.$$

Now a simple principal point is always a semi-principal point, and a perfect principal point is always a compound principal point; but the converse of the above two propositions are not necessarily true. Thus, in

compound principal points, there are two kinds of points, namely, perfect principal point and semi-principal point. Therefore we have the following table of classification of the points of a continuous set.

Table of Classification.

Points of Continuous Set.	{	Non-principal points....	Separating points....	{	Perfectly separating points.
					Imperfectly separating points.....
				
					The first kind.
					The second kind.
	{	Principal points....	Non-separating points.	{	
				
					Simple principal points.
					Compound principal points
					Perfect principal points.
					Semi-principal points.

Part IV.

End Point Defined.

Its Relation to Principal Points.

In this part, we shall define the end point of a continuous set by means of principal points, and investigate the properties of the end point thus defined and its relation to principal points.

Definition 22. If a continuous set of points has a point, such that it is not an interior point of any continuous component having any two points of the set as a pair of principal points, then this point is called an end point of the set.

All the points other than the end point are called inner points of the set.

Theorem 246. When a continuous set has only one pair of principal points (A, B) , its principal points are end points, and the set has no other end point.

Proof. Suppose that one of the principal points (say A) were not an end point, then there would be a component $M_r(C, D)$ containing A as its interior point. But, since any components having (B, C) as well as (B, D) as a pair of principal points cannot contain A by hypothesis, the two components $M_r(C, A)$ and $M_r(D, A)$, contained in $M_r(C, D)$, must contain

a common continuous part M_A containing A (¹). Now take a point P in this part M_A ; then $M_r(C, P)$, a component of $M_r(C, A)$, cannot contain A ; for, if so, $M_r(C, A)$ would have A and P as conjugate principal points of C with respect to $M_r(C, A)$, and accordingly $M(A, B)$ would be a set of the third kind (Theorem 55, Cor.), which contradicts the hypothesis that $M(A, B)$ is a set of the second kind. Similarly $M_r(D, P)$, a component of $M_r(D, A)$, cannot contain A . Thus there arises a contradiction that the sum of $M_r(D, P)$ and $M_r(C, P)$, which must be identical with $M_r(C, D)$, does not contain A . Therefore the principal point is an end point. Further, since any non-principal point is an interior point of $M(A, B)$, so it cannot be an end point. (Q.E.D.)

Cor. If a set has a principal point which is not an end point, then the set is of the third kind.

Theorem 247. If, in a continuous set having an infinite number of pairs of principal points, the number of compound principal points be only one, then it is an end point.

This is proved in a manner similar to the previous theorem.

Cor. If a set has a compound principal point, which is not an end point, then the set is of the third kind and second class.

Theorem 248. A continuous set having more than one compound principal points can have no end point.

Proof. Since the set has more than one compound principal points, the principal points of the set can be divided into two aggregates, such that any element of one aggregate is a conjugate principal point of any one of the other aggregate with respect to the set, and the number of elements of each aggregate is more than one. Thus, when any point P be taken in the set, we can always find a pair of principal points (A_m, B_m) , each of which is different from P . So P is an interior point of $M(A_m, B_m)$, and accordingly cannot be an end point of the set.

Theorem 249. Any continuous set having more than two end points cannot have a pair of principal points.

This theorem follows from Theorems 246, 247, 248, or may be proved directly as follows.

(¹) For, since $M(B, C)$ and $M(B, D)$ does not contain A , so each of them can contain no point in a certain neighborhood of A ; denote by M_A a continuous set containing A in this neighborhood. Then M_A must belong to $M(C, A)$, since the sum of $M(C, A)$ and $M(B, C)$ is identical with the whole set. Similarly it belongs to $M(D, A)$. Therefore M_A is a common continuous part of $M(C, A)$ and $M(D, A)$.

Proof. Denote by E_1, E_2, E_3 the three end points of a continuous set M , and suppose that the set has a pair of principal points (A, B) ; then these points A, B must be end points of the sets, for, if otherwise, the end points E_1, E_2 would be interior points of $M(A, B)$, contrary to the definition of end point. Thus A, B are identical with two of the end points, say E_1, E_2 . But then the third point E_3 will be an interior point of $M(A, B) \equiv M(E_1, E_2)$, which is again contrary to the hypothesis that E_3 is an end point. Therefore the set cannot have a pair of principal points.

Theorem 250. *If a continuous set having only two end points has a pair of principal points, then these end points are so, and the set has no other pair of principal points.*

Proof. That these two end points A, B are a pair of principal points may be proved in the same way as in the previous theorem. Further, if there were another pair of principal points (say C, D), A, B would be interior points of $M(C, D)$, which contradicts the hypothesis that A, B are end points. *Q. E. D.*

It is to be noted that there is a continuous set having only two end points, but no pair of principal points. For example, the set defined by the equations

$$\begin{array}{ll} \text{i)} & -1 \leq x \leq +1, \quad -1 \leq y \leq +1; \\ \text{ii)} & x=0, \quad +1 \leq y \leq +2; \\ \text{iii)} & x=0, \quad -2 \leq y \leq -1, \end{array}$$

is one having the said property.

Theorem 251. *If a continuous set having only one end point has a pair of principal points, then the set has an infinite number of pairs of principal points, and has the end point as only one compound principal point of the set.*

Proof. The end point A must be an element of a pair of principal points, since, if otherwise, it would be an interior point of the set. Denote by B a conjugate principal point of A . If the set had only one pair of principal points, then both of A, B would be end points of the set (Theorem 246), which contradicts the hypothesis that the number of end points is only one. Thus the set has other pairs of principal points, and consequently an infinite number of them.

In this case, the point A is always one element of all these pairs of principal points, since, if otherwise, A would be an interior point of $M(C, D)$, where each of C, D is different from A , and so A could not be an end point of the set, contrary to the hypothesis. Hence A is only

one compound principal point of the set.

Theorem 252. *If a continuous set having no end point has a pair of principal points, then the set has an infinite number of them, and all these principal points are compound.*

Proof. Denote by (A, B) one pair of principal points of the set. Since neither A nor B is an end point, there is another pair of principal points (Theorem 246, Cor.), and accordingly an infinite number of them. Thus at least one of A, B is a compound principal point. But since the set has no end point, the number of compound principal points must be more than one (Theorem 247). Hence it follows from Theorem 25 and 27 that all principal points are compound.

From the above investigations, we find that there exists a very interesting relation between the number of end points and that of compound principal points. It is exhibited in the following table.

number of compound principal points	number of end points.
none	two
one	one
two (infinite)	none

When a continuous set has more than two end points, the set has no pair of principal points.

When a continuous set has more than one compound principal points, the set has no end point.

Moreover from Theorem 28 and the above table, we have the following theorem.

If all the principal points of a set be simple, then the set has two and only two end points; if they be all compound, no end point; if they consist of the both kinds, one and only one end point.

Theorem 253. *If two points A, B , forming a pair of n -ple⁽¹⁾ principal points of a set, be end points of the set, then all of its constituent components are of the second kind, and have common continuous components containing one of the n -ple principal points as its end point.*

(¹) n denotes a finite number.

Proof. Consider any constituent component $M_r(A, B)$. Since all of its components, having any two points other than A (B) as a pair of principal points, cannot contain A (B), so the component $M_r(A, B)$ must be of the second kind.

Further taking any two constituent components $M_r(A, B)$ and $M_s(A, B)$, consider a point C on $M_r(A, B)$ and a point D on $M_s(A, B)$. As the component $M(C, D)$ of the set $M_r(A, C) + M_s(A, D)$ does not contain A , so $M_r(A, C)$ and $M_s(A, D)$ must have a common point E other than A and therefore must contain the components $M_r(A, E)$ and $M_s(A, E)$ respectively. If $M_r(A, E)$ be identical with $M_s(A, E)$, then the constituent components $M_r(A, B)$ and $M_s(A, B)$ have a common continuous component containing A , and the theorem is thus proved.

But if $M_r(A, E)$ be not identical with $M_s(A, E)$, then again take a point C_1 on $M_r(A, E)$ and a point D_1 on $M_s(A, E)$. By the same reasoning as above, it follows that $M_r(A, C_1)$ and $M_s(A, D_1)$ must have a common point E_1 other than A . Further if $M_r(A, E_1)$ be not identical with $M_s(A, E_1)$, then there exists a third common point E_2 and so on. Thus we have a sequence of common points

$$E, E_1, E_2, \dots$$

whose limiting point is A .

Now by the previous proof, $M_r(A, B)$ and $M_s(A, B)$ are both of the second kind. If these sets would have a sequence of common points $\{E\}$, not continuous, in the neighborhood of A , then they would contain an infinite number of components having (A, B) as a pair of principal points (Theorem 196)⁽¹⁾. So (A, B) would be a pair of principal points of an order higher than n , contrary to the hypothesis. Therefore $M_r(A, B)$ and $M_s(A, B)$ have a common continuous component containing A . Since they are any two constituent components, it follows at once that all the constituent components have a common continuous component containing A . The same is true of another n -ple principal point B .

It is clear that A and B are end points of the common continuous components, since, if otherwise, they would also be not the end points of the given set.

Cor. If one and only one of two points A, B , forming a pair of n -ple principal points, be an end point of the set, then the set has a common continuous component containing the end point (say A); and the other point B is a simple principal point of each of the constituent components.

(¹) Theorem 196 is true for any continuous set, namely constituent components of the set may be simple or non-simple, ordinary or semi-singular.

Theorem 254. If, in a continuous set with a pair of n -ple principal points, all of its constituent components be of the second kind, and have two common continuous components, each containing one of the n -ple principal points, then the n -ple principal points are end points of the set.

Proof. Denote by (A, B) the pair of n -ple principal points, and take any two points C, D of the set other than A ; then, since the constituent components are all of the second kind, every one of $M_{t_0}(C, P)$, $M_{t_1}(P, P')$, $M_{t_2}(P', P'')$, \dots , $M_{t_{m-1}}(P^{(m)}, D)$ does not contain A , where $P, P', P'' \dots, P^{(m)}$ are any common points of the constituent components, other than A . Accordingly any component having any two points C, D as a pair of principal points does not contain A , and so, by definition, A is an end point of the set. Similarly it may be proved that B is also an end point of the set.

Theorem 255. If a set with a pair of n -ple principal points has two end points, then the points form a pair of n -ple principal points, and the set has no other pair of principal points of any order whatever.

Proof. Denote by P, Q the two end points of the set, and by (A, B) a pair of n -ple principal points; then P, Q must be identical with A, B . For, if each of P, Q were different from either of A, B , then it would be an interior point of a constituent component $M_r(A, B)$, which contradicts the hypothesis.

Thus in this case both of the n -ple principal points A, B are end points, and so by Theorem 253 all of its constituent components are of the second kind, having a common continuous component containing A and also that containing B . Therefore, by Theorem 240⁽¹⁾, the set has only one pair of n -ple principal points. Further the sum of all the components, which have any two points R, S , other than A (or B), as a pair of principal points, cannot contain A (or B); and accordingly these two points R, S cannot form a pair of principal points of any order whatever.

Cor. 1. If a set with a pair of n -ple principal points has only one end point, then it is an element of the n -ple principal points, and the set cannot have a pair of principal points of any order whatever, without having this point as one of its elements.

Cor. 2. If a set with only one pair of n -ple principal points has two end points, then the set has only one pair of n -ple principal points.

Theorem 256. Any set with only one pair of n -ple principal points cannot have more than two end points.

(¹) Theorem 240 is true for any continuous set, namely constituent components of the set may be simple or non-simple, ordinary or semi-singular.

Proof. If we denote the pair of n -ple principal points by (A, B) , then any end point (if it exist) must be identical with A or B . So there cannot be end points other than A, B .

Theorem 257. Any set with more than one pair of n -ple principal points cannot have an end point unless all the pairs have one element in common.

Proof. Denote two pairs of n -ple principal points by (A, B) and (A', B') . If these four points be different from one another, then any point of the set is an interior point of $M_r(A, B)$ or $M_r(A', B')$ ($r=1, 2, 3, \dots, n$), and so the set has no end point. *Q. E. D.*

Theorem 258. An end point has the following property: Any two continuous components, having an end point P and any other point of the set as a pair of principal points, have a common continuous component containing P , or else they have a set of common points having P as its limiting point.

Proof. Denote by $M_1(P, C_1)$ and $M_2(P, C_2)$ any two continuous components having the said property. If these components have no common point other than P , then C_1, C_2 would be a pair of principal points of $M_1(P, C_1) + M_2(P, C_2)$, and this component would contain P as an interior point of it, contrary to the hypothesis. Thus they have common points other than P ; denote one of them by D_1 , then $M_1(P, D_1)$ and $M_2(P, D_1)$ are or are not identical with each other. In the former case, the two components have a common continuous component containing P . In the latter case, they must have common points other than D_1 . For, if otherwise, describe a very small sphere with P as centre and determine two points Q_1, Q_2 , such that $M_1(P, Q_1)$ and $M_2(P, Q_2)$ lie entirely in the sphere. Then since $M_1(P, Q_1)$ and $M_2(P, Q_2)$ have no common point other than P , the sum of them would have a pair of principal points Q_1, Q_2 , and contain P as its interior point, contrary to the hypothesis. Thus $M_1(P, Q_1)$ and $M_2(P, Q_2)$ have a common point, however small we take the radius of the sphere, therefore P is a limiting point of these common points.

Remark. The converse of the above theorem is not necessarily true, for there is a set containing such a point that it has the above property, yet is not an end point. For example, in a set of points defined by the equations

$$\text{i)} \quad y = \sin \frac{\pi}{1-x}, \quad 0 \leq x < 1$$

$$\text{ii)} \quad y = (-1, +1), \quad x = 1$$

the two points $R_1(1, 1)$, $R_2(1, -1)$ have the said property, though they are not end points.

Definition 23. A point of a continuous set, which has the property stated in Theorem 268, yet is not an end point, is called a pseudo-end point.

Theorem 259. 1. Any continuous set having a pseudo-end point always contains a component of the third kind having the point as a principal point of it.

2. If, in the above component of the third kind, the aggregate of principal points, containing the pseudo-end point, be a continuous set not containing a component of the third kind, then the pseudo-end point is an end point of the aggregate.

Proof. By the property of the pseudo-end point, the set has a continuous component $M(E, F)$ containing the pseudo-end point Q as its interior point; and its components $M(E, Q)$ and $M(F, Q)$ have a set of common points containing Q as their limiting point. Take any one C_k of the common points, then at least one of their components $M(E, C_k)$ and $M(F, C_k)$ must contain Q , and accordingly at least one of $M(E, Q)$ and $M(F, Q)$ is of the third kind having the two points C_k , Q as its conjugate principal points of E or F .

Assume that $M(E, Q)$ is of the third kind, and denote by \mathfrak{M} the aggregate of its conjugate principal points of E . When the aggregate \mathfrak{M} does not contain a component of the third kind, Q is an end point of it. For, if Q were not so, then in \mathfrak{M} there would be a continuous component $M(G, H)$ containing Q as its interior point, and its components $M(G, Q)$ and $M(H, Q)$ would contain a set of common points $\{C'\}$. But \mathfrak{M} having contained no component of the third kind, $M(G, C'_k)$ and $M(H, C'_k)$ do not contain Q while its sum contains $M(G, H)$; which is clearly a contradiction. Thus Q is an end point of the aggregate of principal points.

Theorem 260. Two continuous sets of points are of the same kind when the points of the two sets can be brought into a one-to-one continuous correspondence.

Proof. Denote by M_1, M_2 the two continuous sets, whose points are in a one-to-one continuous correspondence, and suppose that M_1 has a pair of principal points (A_1, B_1) . If A_2, B_2 are two points of M_2 corresponding to A_1, B_1 , then A_2, B_2 are also a pair of principal points of M_2 . For, if this were not the case, then M_2 would contain a proper continuous component containing A_2, B_2 ; denote this component by M'_2 and corresponding one of M_1 by M'_1 . This component M'_1 would then

be a proper continuous component of M_1 containing (A_1, B_1) , contrary to the hypothesis.

Further suppose that (A_1, B_1) is not a pair of principal points of M_1 , then (A_2, B_2) cannot also be a pair of principal points of M_2 . For, if (A_2, B_2) were the pair, then, by the previous proof, their corresponding points A_1, B_1 would also be the same with respect to M_1 , contrary to the hypothesis. *Q. E. D.*

When there is given a continuous set of points, there are many transformations, which transform the given set, so that the points of the transformed one make a one-to-one continuous correspondence with those of the original. Thus from a given continuous set, we get many sets of the same kind, having various forms.

CHAPTER III.

CONTINUOUS SET OF CURVES.

Part I.

Definitions and Fundamental Theorems.

By the fundamental property of simple curves (Jordan curves), the points on two simple curves can be brought into a one-to-one continuous correspondence, and this consideration may be extended to any number of simple curves. Thus we may consider that each curve of a set of simple curves has a one-to-one continuous correspondence with one fixed curve, so that the points which correspond to any one point of the latter curve form a set of points, called *a set of corresponding points*.

Let a set of corresponding points, corresponding to a point P_a of one fixed curve, be denoted by $\{P_{a,u}\}$, where a is a value of v in the interval $(0 \leq v \leq 1)$. When a sequence $(P_{a,u_0}, P_{a,u_1}, \dots, P_{a,u_n}, \dots)$ in $\{P_{a,u}\}$ has $P_{a,q}$ as a limiting point of it, if the same hold of every set of corresponding points, namely the sequence $(P_{v,u_0}, P_{v,u_1}, \dots, P_{v,u_n}, \dots)$ has the point $P_{v,q}$ as a limiting point of it for every value of v , then *the sets of corresponding points are said to be well ordered (or well related) to one another*.

Definition I. A set of simple curves, no one of which intersects with another, is said to be continuous when it is possible to establish a correspondence, such that all sets of corresponding points are continuous and well related to one another.

A continuous set of simple curves defined above may be continuous or discontinuous, considered as a set of points. Thus in order that the above set of curves may also be a continuous set of points, it is necessary to add that it is closed, considered as a set of points. But we shall first investigate the properties of the continuous set of curves in the former (wider) sense; and then that in the latter (narrower) sense.

Definition II. When a continuous set of simple curves has two distinct curves a and b , such that the set has no proper continuous component containing a and b , these two curves are called a pair of principal curves of the set.

Here by a component of a set of curves is meant a system of curves contained in the set, every curve being considered as an indecomposable element.

A continuous set of curves with a pair of principal curves a, b is denoted by $\mathfrak{M}(a, b)$, and any element other than principal curves is called an interior curve of the set.

Theorem I. If c and \bar{c} are any two curves of a continuous set of curves, then there is a continuous component having c, \bar{c} as a pair of principal curves.

This may be proved in exactly the same manner as in the case of continuous set of points.

Theorem II. In a set of corresponding points of a continuous set of curves $\mathfrak{M}(a, b)$, the two points of the set, which correspond to a pair of principal curves a, b , form a pair of principal points of the set of corresponding points.

Proof. Denoting by A_v, B_v the two points of a set of corresponding points $\{P_v\}$, corresponding to the principal curves a, b , we shall prove that (A_v, B_v) is a pair of principal points of the set $\{P_v\}$.

Since $\{P_v\}$ is a continuous set of points containing A_v, B_v , it contains a continuous component having A_v, B_v as a pair of principal points; denote it by $M(A_v, B_v)$. If $\{P_v\}$ were not identical with $M(A_v, B_v)$, then there would be certain points of $\{P_v\}$ not contained in $M(A_v, B_v)$. Denote the set of them by $\{P'_v\}$, and the set of corresponding curves by $\{p'\}$, then the set of curves $[\mathfrak{B}(a, b) - \{p'\}]$ must be continuous as will be proved below.

In the first place, any set of corresponding points $[\{P_{v_a}\} - \{P'_{v_a}\}]$ is closed, since $[\{P_v\} - \{P'_v\}] \equiv M(A_v, B_v)$ is closed and well related to $[\{P_{v_a}\} - \{P'_{v_a}\}]$.

Next the set $[\{P_{v_a}\} - \{P'_{v_a}\}]$ is connected. For, suppose that it

were not so, then it might be divided into two parts $M_{e_a}^{(1)}, M_{e_a}^{(2)}$, such that the distance between any point of the one part and that of the other is greater than a finite number ε . Thus any point of the one part cannot be a limiting point of the other. Since all the sets of corresponding points are well related, the same must be true of the two corresponding parts $M_v^{(1)}, M_v^{(2)}$ of $[\{P_v\} - \{P_v'\}]$. Therefore both of them must be closed, the set $[\{P_v\} - \{P_v'\}] \equiv M_v^{(1)} + M_v^{(2)} \equiv M(A_v, B_v)$ being continuous. Hence it follows that the sum of two closed sets $M_v^{(1)}, M_v^{(2)}$, having no common point, is continuous, which is clearly impossible. So the set $[\{P_{e_a}\} - \{P'_{e_a}\}]$ is connected and closed, namely continuous.

From the continuity of the sets of corresponding points, and their being well related, we conclude that the set of curves $[\mathfrak{M}(a, b) - \{p'\}]$ is continuous.

Now the set $[\mathfrak{M}(a, b) - \{p'\}]$ clearly contains the elements a, b ; and thus the supposition that $\{P_v\}$ were not identical with $M(A_v, B_v)$ led us to the result that a proper continuous component of $\mathfrak{M}(a, b)$ would contain (a, b) , contrary to the definition of $\mathfrak{M}(a, b)$. Therefore $\{P_v\}$ must be identical with $M(A_v, B_v)$.

Theorem III. If a component $\{p\}$ of a continuous set of curve $\mathfrak{M}(a, b)$, is a corresponding one to a continuous component $M(E_v, F_v)$ of a set of corresponding points $M(A_v, B_v)$, then the two curves of the component, corresponding to a pair of principal points (E_v, F_v) , form also a pair of principal curves of the component $\{p\}$.

Proof. From the continuity of $M(E_v, F_v)$, it may be proved that the corresponding component $\{p\}$ is also a continuous set of curves, in exactly the same manner as in the previous theorem. Since $\{p\}$ contains two curves e, f , corresponding to E_v, F_v , it contains a continuous set of curves having e, f as a pair of principal curves (Theorem I); denote it by $\mathfrak{M}(e, f)$. Now we have to prove that $\mathfrak{M}(e, f)$ is identical with $\{p\}$.

Suppose that this were not the case, then $\{p\}$ would contain certain curves other than those of $\mathfrak{M}(e, f)$, denote the set of them by $\{p'\}$, and a set of corresponding points by $\{P_v'\}$. Since $\mathfrak{M}(e, f) \equiv [\{p\} - \{p'\}]$ is a continuous set of curves having e, f as a pair of principal curves so $M(E_v, F_v) - \{P_v'\}$ must also be a set of corresponding points having E_v, F_v as a pair of principal points (Theorem II), which contradicts the fact that $M(E_v, F_v)$ is a continuous set having E_v, F_v as a pair of principal points. Therefore $\mathfrak{M}(a, b)$ is identical with $\{p\}$.

Theorem IV. The elements of a continuous set of curves $\mathfrak{M}(a, b)$ are

in a one-to-one continuous correspondence with the points of any set of corresponding points.

Proof. Consider two aggregates of principal points of all the elements (Jordan curves) of $\mathfrak{M}(a, b)$, and denote them by $\{R_p\}$ and $\{S_p\}$; then it is clear that these aggregates must always be the sets of corresponding points for any correspondence whatever. By the definition of continuous set of curves, $\mathfrak{M}(a, b)$ has a correspondence I , such that all sets of corresponding points are continuous and well related to one another. Now in the above correspondence, take any set of corresponding points and denote it by $\{P_p\}$; then $\{P_p\}$ and $\{R_p\}$ are both continuous and well related by the continuity of $\mathfrak{M}(a, b)$. Hence following the method of reasoning used in Theorem II, we may prove that any component of $\{P_p\}$, corresponding to any continuous component of $\{R_p\}$, is also continuous.

After having proved a relation between continuous components of $\{P_p\}$ and $\{R_p\}$, we shall proceed to prove that $\{P'_p\}$, a component of $\{P_p\}$, corresponding to a continuous component $\{p'\}$ of $\mathfrak{M}(a, b)$, is also continuous. Since $\{p'\}$ is a continuous set of curves, it has a correspondence I' such that all sets of corresponding points are continuous. If I' be identical with I , then the proposition is proved at once; but as we cannot know whether I' is identical with I or not, we must proceed as follows. Denote by $\{R'_p\}$ a component of $\{R_p\}$ corresponding to $\{p'\}$, then since $\{R'_p\}$ is a set of corresponding points for any correspondence, so it is continuous by the continuity of $\{p'\}$. But, by what has just been proved, the component of $\{P_p\}$ corresponding to a continuous component of $\{R_p\}$ is continuous, and therefore the component $\{P'_p\}$ corresponding to a continuous component $\{R'_p\}$ must be continuous. Thus the component of a set of corresponding points, which corresponds to any continuous component of $\mathfrak{M}(a, b)$, is always continuous.

Conversely that the component of $\mathfrak{M}(a, b)$, which corresponds to any continuous component of a set of corresponding points, is also continuous may be proved in exactly the same manner as in Theorem II.

By the definition of corresponding points, the sets $\{P_p\}$ and $\mathfrak{M}(a, b)$ have a one-to-one correspondence, and so from the above discussion they have a one-to-one continuous correspondence.

Having proved the four fundamental theorems, we proceed to give the definitions of certain systems of curves, or elements of the system, which are frequently used in the following discussion.

Definition III. When two curves of a continuous set of curves

form a pair of principal curves, one of them is said to be a *conjugate principal curve* of the other with respect to the set.

Definition IV. If a discontinuous set of curves always contains a continuous component having any two curves of the set as a pair of principal curves, then *the set is said to be semi-continuous*.

Definition V. When a curve of a continuous set of curves is not a conjugate principal curve of any other curve of the set, the curve is called a *non-principal curve of the set*.

All curves of the above set, other than non-principal curves, are called *principal curves of the set*. Hereafter we shall see that principal curves of a continuous set may be divided into two classes, namely that the one having only one conjugate principal curve, and the other having an infinite number of them.

Definition VI. When a curve of a continuous set of curves has only one conjugate principal curve, it is called a *simple principal curve*.

Definition VII. When a curve of a continuous set of curves has an infinite number of conjugate principal curves, it is called a *compound principal curve*.

Definition VIII. When a curve of a continuous set of curves is a conjugate of two principal curves, which form a pair of principal curves of the set, it is called a *perfect principal curve with respect to that pair of principal curves*.

Definition IX. When a curve of a continuous set of curves is a conjugate of one and only one of two principal curves, which form a pair of principal curves of the set, it is called a *semi-principal curve with respect to that pair of principal curves*.

As in the case of set of points, it may be proved that, when a curve of a continuous set of curves $\mathfrak{M}(a, b)$ is neither a conjugate principal curve of a nor that of b , it is a non-principal curve of the set.

Thus elements of a continuous set of curves are divided into the following three kinds: *non-principal curve*, *simple principal curve* and *compound principal curve*; and when referred to a definite pair of principal curves, they are divided into another three kinds: *non-principal curve*, *semi-principal curve*, and *perfect principal curve*.

Remark. 1. A perfect principal curve is always a compound principal curve, but the converse is not necessarily true.

2. simple principal curve is always a semi-principal curve, but the converse is not necessarily true.

Definition X. When a non-principal curve separates a continuous set of curves into two components, two cases may occur, i.e.,

- i) two components separated have no common curve other than the separating curve itself;
- ii) two components separated have common curves other than separating one itself.

In the case i), the separating curve is called a *perfectly separating curve*, and in the case ii) an *imperfectly separating curve*.

A principal curve does not separate the set into two part in ordinary sense, so it may be called a *non-separating curve*.

Definition XI. A continuous set of curves having no non-principal curves is called a *singular set of curves*; and a continuous set containing a proper component having such a property is called a *semi-singular set of curves*; and all other sets the *ordinary ones*.

Definition XII. When a continuous set of curves contains one and only one component having any two curves of the set as a pair of principal curves, it is called a *simple set of curves*.

Definition XIII. When a continuous set of curves has two distinct curves a, b , such that there are n and only n continuous components having (a, b) as a pair of principal curves, and the sum of these components is the set itself, two curves a, b are called a *pair of principal curves of the n^{th} order*, or simply n -ple principal curves.

The continuous set of the n^{th} order, and also that of the n^{th} degree will be defined later.

Definition XIV. If a continuous set of curves has a curve, such that it is not an interior curve of any continuous component having any two curves of the set as a pair of principal curves, then this curve is called an *end curve of the set*.

The curves other than end curves are called *inner curves of the set*.

The definitions given above are nothing but those which are obtained by substituting "curves" for "points" in the definitions given in the set of points. But since it may serve as a summary of definitions given scattered over the previous two chapters, we gave the principal ones of them here.

Properties of Continuous Set of Curves.

By the four fundamental theorems and the previous definitions, we may prove, in a very simple manner, that all the theorems obtained by substituting "curve" instead of "point" in those of continuous set of

points (except certain theorems concerning Jordan curves, Part V)(¹) are also true in the case of a continuous set of curves.

The essentiality of the method of proving them consists of three steps, namely

(1) that, by the four fundamental theorems, from the hypothesis H_C of a given theorem concerning the continuous set of curves is deduced a proposition H_P concerning the set of corresponding points, which is obtained by substituting "point" instead of "curve" in the above hypothesis H_C ;

(2) that, by the theorems concerning the set of points already established in the previous chapters, from the above proposition H_P is deduced a conclusion C_P concerning the set of corresponding points, which is again obtained by substituting "point" instead of "curve" in the conclusion C_C of the given theorem;

(3) that, from this conclusion C_P , the required conclusion C_C of the given theorem is again deduced by the four fundamental theorems.

We shall explain the above by taking an example. Now let us prove the fundamental theorem;

"when a continuous set of curves has two pairs of principal curves, it has an infinite number of them,"

as an example.

Denote the continuous set of curves by \mathfrak{M} , and its two pairs of principal curves by (a_0, b_0) , (a_1, b_1) , and consider a set of corresponding points M_v .

(1) By the fundamental theorem II, two pairs of points $(A_{0,v}, B_{0,v})$, $(A_{1,v}, B_{1,v})$ of the set of corresponding points M_v , which correspond to (a_0, b_0) , (a_1, b_1) respectively are also pairs of principal points of the set M_v .

(2) Since M_v has two pairs of principal points, it has an infinite number of them by Theorem 4 already established in the case of the set of points.

(3) By the fundamental theorem III, pairs of curves corresponding to the pairs of principal points of the set M_v are also pairs of principal curves of the set of curves \mathfrak{M} ; so from (2) the set of curves has an infinite number of pairs of principal curves. *Q. E. D.*

Thus we have a very important theorem.

(1) If we define simple surface in a similar manner to the case of simple curve, taking continuous set of curves in wider sense, the simple surface thus defined is not necessarily a Jordan surface. More precisely we shall discuss this point later.

Theorem V. All the theorems which are true in the case of continuous set of points are also true in the continuous set of curves, that is to say, from Theorems 1, 2, 3,, 260 established for the set of points, we have the corresponding Theorems I*, II*, III*,, CCLX* holding true for the set of curves, (except certain theorems concerning a Jordan curve).

So we obtain a remarkable result that our continuous set of curves has exactly the same properties as those of continuous set of points.

Classification of Continuous Sets of Curves.

From the property of the continuous set of curves established in Theorem IV*, we may divide the sets of curves into three kinds, i.e.,

- (i) that which has no pair of principal curves ;
- (ii) that which has only one pair of principal curves ;
- (iii) that which has an infinite number of pairs of principal curves.

We shall call (i), (ii), (iii) a continuous set of curves of the first, the second and the third kind respectively, following the terminology used for the set of points.

By Theorem IV*, Cor., when a curve of continuous set of curves has two conjugate principal curves, it has an infinite number of them. Thus principal curves may be divided into two classes, namely the one having only one conjugate principal curve, and the other having an infinite number of them ; the former being called a simple principal curve, and the latter a compound principal curve as was already defined. The properties of simple and compound principal curves are given in Theorems XXIV*—XXVIII*, one of which is the following.

If a continuous set of curves have two compound principal curves, then it has an infinite number of them.

Thus continuous sets of curves having principal curves may be divided into three kinds, i.e.,

- (i) that which has no compound principle curve ;
- (ii) that which has only one compound principl curve ;
- (iii) that which has an infinite number of compound principal curves.

And the relation between the number of compound principal curves and the kinds of principal curves may be stated as follows.

The necessary and sufficient condition that a continuous set of curves should have only compound principal curves is that it should have two compound principal curves.

The necessary and sufficient condition that a continuous set of curves

should have *both kinds of principal curves* is that it should have *only one* compound principal curve.

The necessary and sufficient condition that a continuous set of curves should have *only simple principal curves* is that it should have *no* compound principal curve.

From these properties, we may again classify the continuous sets of curves as follows.

	number of pairs of principal curves.	number of compound principal curves.
Set of the first kind	none	
Set of the second kind	only one	none
Set of the third kind	infinite	$\begin{cases} \text{(i) only one} \\ \text{(ii) infinite} \end{cases}$

Thus the sets of the third kind may again be divided into two sub-classes according as the number of compound principal curves is only one or infinite. We distinguish them as the set of *the first and the second classes* respectively.

By the definition of non-principal curve, we know that the elements of the continuous set of curves may be divided into three kinds, namely:

- (i) non-principal curve,
- (ii) simple principal curve,
- (iii) compound principal curve.

And by the property of non-principal curves that a continuous set of curves has always an infinite number of non-principal curves when it has one of them, we may divide the sets of continuous curves into two kinds, i.e.,

- (i) *that which has no non-principal curve,*
- (ii) *that which has an infinite number of non-principal curves.*

Now it is clear that all curves of any set of the first kind are non-principal curves, and by Theorem XXXIV*, all curves of any singular set are compound principal ones. The other continuous sets contain at least two kinds of curves. From this standpoint also, we may classify the sets of curves. For this purpose, we give a table of the number of curves of three kinds contained in the sets of various kinds.

	number of non-principal curves	number of simple principal curves	number of compound principal curves
Set of the first kind	infinite	none	none
Set of the second kind	infinite	two	none
Set of the third kind and first class	infinite	infinite	one
Set of the third kind and second class	$\begin{cases} \text{(i) infinite} \\ \text{(ii) none} \end{cases}$	none	infinite

Thus the sets of the third kind and second class may again be divided into two subclasses according as they have no non-principal curve or an infinite number of them. We distinguish them as the sets of the *first and second species* respectively.

Or from another point of view, we may classify the continuous sets as follows.

- (I) Set containing *only one kind* of curves $\begin{cases} \text{i) only non-principal curves,} \\ \text{ii) only compound principal curves.} \end{cases}$
- (II) Set containing *two kinds* of curves $\begin{cases} \text{i) simple and non-principal curves.} \\ \text{ii) compound and non-principal curves.} \end{cases}$

(III) Set containing *all (three) kinds* of curves.

As in the continuous sets of points, there is neither continuous set of curves containing only simple principal curves, nor that containing simple and compound principal curves and only these.

The sets of the first kind may be subdivided into two classes according as they have or have not a pair of principal curves of a certain order.

As in the continuous set of points, the set of curves may have pairs of principal curves of different orders. If we define such a set as a *set of the n^{th} order*, when the number of the lowest order of principal curves of the set is n , then the former class of the sets of the first kind may again be classified by the order of principal curves.

Further, as in the set of points, among the continuous sets of curves having no pair of principal curves of any order whatever, there is such

a set, that it contain proper continuous components having pairs of the first, the second,, the n^{th} order, but not that of the $(n+1)^{\text{th}}$ order. If we define such a set as *a set of the n^{th} degree*, then the latter class of the sets of the first kind may be again classified by the degree of them.

Thus we have the following table of classification, which is exactly the same as that of sets of points.

Table of Classification.

Continuous Set of Curves.	Set of the first kind..... (having no pair of principal curves.)	{	The first class (having no pair of principal curves of a certain order.)	{	The first degree. The second degree. The n^{th} degree. (n may be infinite).
	Set of the second kind (having only one pair of principal curves.)	{	The second class (having a pair of principal curves of a certain order.)	{	The second order. The third order. The n^{th} order. (n may be infinite).
	Set of the third kind..... (having an infinite number of pairs of principal curves.)	{	The first class (having only one compound principal curve.)	{	The first species. (Singular set). (having no non-principal curve.)
			The second class (having an infinite number of compound principal curves.)		The second species. (having an infinite number of non-principal curves.)

Part II.

Surface.

In the previous part, we have defined the continuous set of curves in wider sense, and remarked there that the continuous set of curves thus defined is not necessarily continuous, considered as a set of points. Here let us consider a continuous set of curves in narrower sense, namely that which is also continuous, considered as a set of points. It is defined thus:

Definition (A). A set of simple curves, no one of which intersects with another, is said to be continuous when it is closed, considered as a set of points, and moreover it is possible to establish a correspondence, such that all sets of corresponding points are continuous and well related to one another.

In this new definition, though the fundamental Theorem I of the previous part is true, the other three theorems are not necessarily true; namely there is a continuous set of curves, in which we are not able to establish a one-to-one continuous correspondence between the set of curves and a set of its corresponding points. For example, take a set of curves defined by the equations

$$\begin{aligned} \text{(i)} \quad & \begin{cases} z = \frac{2xy}{x^2 + y^2}, & 0 \leq x \leq 1 \quad (0 < y \leq 1) \\ z = 0, & 0 \leq x \leq 1 \quad (y = 0) \end{cases} \\ \text{(ii)} \quad & y = 0, \quad 0 \leq x \leq 1 \quad (0 < z \leq 1), \end{aligned}$$

and denote by $c_{y=m}$ a curve corresponding to a definite value m of y in the equation (i), and by $c_{z=n}$ a similar one in the equation (ii). Now this set is clearly continuous, satisfying the conditions given in Definition (A). Any curve of $\{c_z\}$ is a conjugate principal curve of $c_{y=1}$ with respect to the given set, for, any proper component of the set containing $c_{y=1}$ and $c_{z=m}$ cannot be continuous, owing to the fact that the set $\{c_y\}$, considered as a set of points, is not continuous as it does not contain its limiting points $\{x=0, y=0, z(0 < z \leq 1)\}$. Thus the set may be denoted by

$$\mathfrak{M}(c_{y=1}, c_{z=m}), \quad 0 \leq m \leq 1.$$

Now consider a set of corresponding points $\{P_v\}$ defined by the equations

$$(i) \quad x=v, \quad 0 \leq y \leq 1, \quad z = \frac{2vy}{v^2+y^2};$$

$$(ii) \quad x=v, \quad y=0, \quad 0 < z \leq 1,$$

and denote by $P_{v, y=1}$, $P_{v, z=m}$ those elements of $\{P_v\}$ corresponding to $c_{y=1}$, $c_{z=m}$, then, though $(c_{y=1}, c_{z=m})$ is a pair of principal curves of $M(c_{y=1}, c_{z=m})$, yet their corresponding points $P_{v, y=1}$, $P_{v, z=m}$ do not form a pair of principal points of $\{P_v\}$ when $m \neq 1$. Thus the fundamental Theorem II is not true in this set of curves.

Further take a continuous component of $\{P_v\}$ defined by the equations

$$(i) \quad x=v, \quad 0 \leq y \leq 1, \quad z = \frac{2vy}{v^2+y^2};$$

$$(ii) \quad x=v, \quad y=0, \quad 0 < z \leq \frac{1}{2},$$

then the points $A \left(x=v, y=1, z=\frac{2v}{v^2+1} \right)$, $B \left(x=v, y=0, z=\frac{1}{2} \right)$ form a pair of principal points of it, while the corresponding component of curves is not continuous as it does not contain its limiting points $\left\{ x=0, y=0, \frac{1}{2} < z \leq 1 \right\}$. Thus the fundamental Theorem III is not true in this set of curves, and so also the fundamental Theorem IV.

But there are systems of continuous sets of curves satisfying the conditions given by Definition (A), for which the four fundamental theorems and accordingly all theorems (I*—CCLX*) are also true. We give here an example of them.

Consider any plane continuous set of points $\{P\}$, and at every point of it, erect a perpendicular of the length 1 to the plane, then this set of straight lines is continuous according to Definition (A). For, if we define the set of points, which are in the same distance from the plane, as a set of corresponding points, then any set of corresponding points is nothing but the plane set of points transferred parallel to itself in the direction perpendicular to the plane, and therefore is continuous and well related to one another. Further take any sequence of points $(Q_1, Q_2, \dots, Q_n, \dots)$ on the set of straight lines, and denote by (x_n, y_n, z_n) the co-ordinates of the point Q_n , taking the given plane as plane of x, y , and a perpendicular to it as axis of z . Denote by z the limiting value of the sequence $(z_1, z_2, \dots, z_n, \dots)$ and by (x, y) that of the sequence $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), \dots\}$; then the point having (x, y, z) as its co-

ordinates is clearly the limiting point of the sequence $(Q_1, Q_2, \dots, Q_n, \dots)$. Now the set $\{P\}$ being a continuous one of points, and the points of the sequence $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), \dots\}$ being elements of $\{P\}$, the limiting point of the sequence must belong to $\{P\}$, denote it by P_i . Also the limiting value z_i of $(z_1, z_2, \dots, z_n, \dots)$ is included between 0 and 1. Thus the limiting point Q of the sequence $(Q_1, Q_2, \dots, Q_n, \dots)$ clearly lies on the perpendicular erected at the point P_i of the given plane, and so it belongs to the set of the straight lines considered. Therefore the set of straight lines is continuous in the narrower sense.

In this set of straight lines, since any component corresponding to any continuous component of the plane set of points is always continuous as was proved above, it follows at once that the fundamental theorems I, II, III, IV are all true in this set of lines.

Here we shall discuss the most important ones of the continuous sets of curves in a narrower sense, namely the ones called simple surfaces (open and closed).

[I]

Definition A. A continuous set of simple curves is said to be a simple surface when any continuous component of it is of the second kind.

In the Memoirs referred to at the commencement of this paper I have proved the following theorems concerning the simple surface.

Theorem A₁. The simple surface is identical with a Jordan surface.

Theorem A₂. The set of corresponding points in the simple surface is a Jordan curve.

Theorem A₃. Any two elements of the set determine one and only one Jordan surface.

By exactly the same reasoning as in the continuous set of points, we may prove the following theorem.

A continuous set of simple curves, which is of the second kind and has no component of the third kind, is a Jordan surface; and conversely a Jordan surface is a set of the second kind having no component of the third kind.

From this theorem we may prove that the surfaces defined in the following definitions are all identical with a Jordan surface. The method of proving them is similar to that for the continuous set of points.

The first definition. A continuous set of curves is said to be a simple surface when any continuous component of it has one and only one pair of principal curves.

The second definition. A continuous set of curves is said to be a simple surface when it satisfies the following conditions :

- (i) it has a pair of principal curves,
- (ii) any interior curve of it is a perfectly separating curve of the set.

The third definition. A continuous set of curves is said to be a simple surface when it satisfies the following conditions :

- (i) it has a pair of principal curves,
- (ii) when any two different pairs of curves of the set are taken, the continuous components having these pairs as those of principal curves are also different.

The fourth definition. A continuous set of curves is said to be a simple surface when it satisfies the following conditions :

- (i) it has a pair of principal curves,
- (ii) any two curves of the set determine a continuous component having them as only one pair of principal curves.

The fifth definition. A continuous set of curves is said to be a simple surface when it satisfies the following conditions :

- (i) it has a pair of principal curves a, b ,
- (ii) when c is any interior curve of the set and d that of the component $M(a, c)$ (or $M(b, c)$), the set has always a continuous component containing d and a (or b), but not c .

The sixth definition. A continuous set of curves is said to be a simple surface when it satisfies the following condition :

if any three curves be taken in the set, then one and only one of them is an interior curve of a continuous component having the other two as a pair of principal curves.

[II]

Now let us consider the second important continuous set of curves defined by the following

Definition \mathfrak{B}_1 . A continuous set of simple curves, which is of the first kind, is said to be a closed simple surface, when all of its proper continuous components are of the second kind.

The following theorems concerning the closed simple surface may be proved in a similar manner as in the case of the continuous set of points.

Theorem B_1 . The set defined above is identical with a closed Jordan surface.

Theorem B_2 . The set of corresponding points of a closed simple sur-

face is a closed Jordan curve.

Theorem B_3 . Any two elements of the set determine two and only two Jordan surfaces, such that the sum of them is the set itself.

If we define the continuity of the set of closed simple curves in a similar manner as in the case of continuous set of simple curves (in a narrower sense), the closed simple surface may be considered as a continuous set of closed simple curves, and may be defined thus:

Definition B_2 . A continuous set of closed simple curves is said to be a closed simple surface, when any continuous component is of the second kind.

Of the closed surface thus defined, we have the following theorems corresponding to Theorems B_2 and B_3 , which rather resemble Theorems A_2 and A_3 .

Theorem B_2' . The set of corresponding points of a closed simple surface is a Jordan curve.

Theorem B_3' . Any two elements of the set determine one and only one closed Jordan surface.

[III]

In the case [I], the element of the continuous set and the set of corresponding points are both simple curves; and in the case [II], one of them is a simple curve while the other a closed simple curve. Thus there still remains a third kind of the set, namely a set whose element and whose set of corresponding points are both simple closed curves.

Definition G . A continuous set of closed simple curves, which is of the first kind, is said to be a total closed surface (ring) when all of its proper continuous components are of the second kind.

The following theorems concerning a total closed surface may be proved in a similar manner as in the case [II].

Theorem C_2 . The set of the corresponding points of a total closed surface is a closed Jordan curve.

Theorem C_3 . Any two elements of the set determine two and only two closed Jordan surfaces, such that the sum of them is the set itself.

CHAPTER IV.

CONTINUOUS SET OF SURFACES.

CONTINUOUS SET OF R -DIMENSIONAL SIMPLE FIGURES
IN N -DIMENSIONAL CONTINUUM.

In the Memoirs referred to at the commencement of this paper we have proved that a one-to-one continuous correspondence may be established between the points on the simple surface and the system of values μ, ν in the domain $D[0 \leq \mu \leq 1, 0 \leq \nu \leq 1]$. By this property, the points on two simple surfaces can be brought into a one-to-one continuous correspondence, and this consideration may be extended to any number of simple surfaces. Thus we may consider that each surface of a set of surfaces has a one-to-one continuous correspondence with one fixed surface, so that the points which correspond to any one of the latter surface form a set of points, called a set of corresponding points.

Definition (i). A set of simple surfaces, no one of which intersects with another, is said to be continuous when it is possible to establish a correspondence, such that all sets of corresponding points are continuous and well related to one another.

Definition (ii). When a continuous set of simple surfaces has two distinct elements a, b , such that the set has no proper continuous component containing a, b , these elements a, b are called a pair of principal surfaces.

A continuous set of surfaces with a pair of principal surfaces a, b is denoted by $\mathfrak{M}(a, b)$, and any other element is called an interior surface of the set.

From these definitions we may prove the following four fundamental theorems, in exactly the same manner as in the case of set of simple curves.

Theorem (i). If c, d be any two elements of a continuous set of surfaces, then there is a continuous component having c, d as a pair of principal surfaces.

Theorem (ii). In a set of corresponding points of a continuous set of surfaces $\mathfrak{M}(a, b)$, two points A, B of the set, corresponding to a pair of principal surfaces a, b , are also a pair of principal points of the set of corresponding points.

Theorem (iii). If a component $\{p\}$ of a continuous set of surfaces $\mathfrak{M}(a, b)$ is a corresponding one to a continuous component $M(E_n, F_r)$ of a

set of corresponding points $M(A_v, B_v)$, then the two surfaces of the component corresponding to a pair of principal points E_v, F_v form also a pair of principal surfaces of the component $\{p\}$.

Theorem (iv). The elements of a continuous set of surfaces $\mathfrak{M}(a, b)$ are in a one-to-one continuous correspondence with the points of any set of corresponding points.

From these theorems, and from the definitions of conjugate principal surface, simple and compound principal surfaces, non-principal surface, singular system of surfaces, etc., corresponding to those of the set of points and the set of curves, we may prove that all the theorems in the set of points (except certain theorems⁽¹⁾ regarding to Jordan curve) hold also in the continuous set of surfaces, and accordingly we may classify the continuous sets of surfaces as in the case of sets of points and sets of curves.

Of course this idea, without difficulty, may be extended to a continuous set of r -dimensional simple figures⁽²⁾ in n -dimensional continuum. Thus finally we have a theorem in a very general form.

Theorem. The continuous sets of r -dimensional simple figures in n -dimensional continuum have the same properties as the continuous sets of points, and accordingly they may be classified as in the case of sets of points; that is to say, the 260 theorems and tables of classification, given in Chapters I and II, when the word " r -dimensional simple figures" is substituted for "points," are also true in the continuous sets of r -dimensional simple figures.

Remark. There may arise a question whether all kinds of continuous sets of r -dimensional simple figures may be conceived. It will be seen that it depends on the dimensions of the space in which the continuous set lies. For example, in two dimensional space, all kinds of the continuous sets of points may be conceived, while it is not the case with the continuous sets of curves, namely, though, in this space, it is possible to conceive certain sets of curves belonging to the second kind, yet it is impossible to conceive any set of the third kind. In three dimensional space, all kinds of continuous sets of curves may be conceived, but it is not so with the continuous sets of surfaces.

Thus, in general, we see that it is always possible to conceive any

(¹) The theorems regarding to Jordan figure hold for certain continuous set of surfaces in the narrower sense.

(²) The definition and the fundamental properties of r -dimensional simple figures were given in the Memoirs referred to.

kind of continuous set of r -dimensional simple figures in n -dimensional continuum when $r \leq n-2$, taking the set of corresponding points in two dimensional continuum.

(The End.)

抄 録 短 評

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當ナル挿入ヲ始メトシ、極メテ適切ナル題材選擇配列ニ勉メ、最良ノ教科書トシテ推稱セラルベシ、大數學者ノ肖像ヲ挿入シタル教科書ハ他ニ類ヲ見出サザルガ如シ。其等ノ學者ニハ Wallis, Hamilton, Newton, Descartes, Vieta 及 Gauss ガ選定セラレタリ。(T. H.).

II. 雜 誌 内 容

下記ノ雜誌ニ掲載セラレタル論文中、數學又ハ數理物理學ニ關係ナキモノハ省略ス。

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雜 錄 彙 報

定幅曲線ニ關スル極大極小問題ニツイテ

本誌第十一卷第 92 頁ニ於テ、余ハ掛谷君ト共著ノ論文 On some problems of maxima and minima for the curve of constant breadth and the in-revolvable curve of the equilateral triangle 中ニ、正方形ノ内轉形即チ定幅曲線ノ切點ト正方形ノ角點トノ間ノ距離ガ極大極小トナリ得ルモノハ Reuleaux ノ定幅曲線ナルコトヲ、幾何學的ニ證明シタ。此ノ問題ヲ解析的ニ言ヒ表セバ、

$$\int_0^{\pi} \rho(\theta) \sin \theta d\theta = h, \quad \int_0^{\pi} \rho(\theta) \cos \theta d\theta = 0, \quad 0 \leq \rho(\theta) \leq h$$

ナル制限ノ下ニ、 $\int_0^{\frac{\pi}{2}} \rho(\theta) \sin \theta d\theta$ ノ極大極小ヲ求ムル問題トナル。之ハ積分方程式ニ關スル

掛谷君ノ理論ニ包含セラレルガ、具體的ノ解答ハ解析的ニ得ラレナカツタ。然ルニ今マデ余ノ注意ヲ逸シテ居タ Markoff ノ重要ナル論文 Recherches sur les valeurs extrêmes des intégrales et sur l'interpolation, Acta mathematica 28, 1904 ヲ見ルニ及ンデ、カヽル種類ノ極大極小問題ハ早く已ニ Markoff ノ論ゼシ所ノモノデアツテ、上ノ幾何學上ノ問題ハ Markoff ノ方法ヲ以テスレバ具體的ノ結果ガ得ラルルコトガ分ツタ。Markoff ノ問題トハ

$$\int_a^b f(x) \varphi_k(x) dx = C_k, \quad (k=1, 2, \dots, n), \quad a \leq f(x) \leq \beta$$

ノ下ニ、 $\int_a^t f(x) \varphi(x) dx \dots (a < t \leq b)$

ノ極大極小ヲ求ムルモノデアツテ、彼ハ之ガ解キ得ル爲メノツノ十分條件ヲ出シ、ソレガ満足セラル、場合ニソノ解答ヲ求ムル方法ヲ與ヘテオル。

コノ方面カラスルト新ニ次ノ結果ガ得ラレル：

定幅曲線ガ二平行切線ニヨリテ分タル、二ツノ部分ノ長サノ差ガ極大トナルモノハ

Reuleaux 曲線デアル。

余ガ彙ニ本誌第三卷第 129 頁ニ於テ、Über die Polynome von der kleinsten totalen Schwankung ト題シテ論ジタ極大極小問題ハ、ソレ以前ニ Korkine, Zolotareff 及ビ Stieltjes ガ論ジタ問題デアツテソノ擴張ガ又上ノ Markoff ノ論文中ニ論ゼラレタアル。Korkine, Zolotareff 氏ノ論文ノアルコトハ、ソノ當時 Kryloff 氏が私信ニヨリテ注意セラレタガ、兩氏ノ論文ニハ second variation ノコトガ論ジテナイ點ニ於テ余ノ論文ト相違シテ居タ爲メソノ儘ニシテオイタノデアツタ。(M. F.)

初等幾何學ノ一問題

一平面上ニ四双ノ點 $A_1, A_2; B_1, B_2; C_1, C_2; D_1, D_2$ ガ與ヘラル、トキソノ平面上ニ一雙ノ點 X_1, X_2 チバ $A_1, A_2; X_1, X_2; B_1, B_2; X_1, X_2; C_1, C_2; X_1, X_2; D_1, D_2; X_1, X_2$ ナル四組ノ四點ガ共圓點ナル様ニ決定セヨ。

本作圖題ハ空間ニ於ケル作圖ヲ許ストキハ次ギノ如ク容易ニ解決セラルベシ。空間ニ於テ一ツノ球ヲ作リソノ中心ヨリ平面ヘ引ケル垂線ガ球ト交ハル點 O ヨリ平面上ノ圖形ヲ球面上ニ射影シ、 $a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2$ ヲ夫レ々 $A_1, A_2; B_1, B_2; C_1, C_2; D_1, D_2$ ヲ射影シテ得タル點トシ、 $a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2$ ヲ結び付ケ、此四直線ニ交ハル所ノ二直線ヲ引キ、ソレガ球ト交ハル所ノ點ヲ夫レ々 $x_1, x_2; x_1', x_2'$ トシ、 $x_1, x_2; x_1', x_2'$ ヲ O ヨリ平面ニ射影シ、 $X_1, X_2; X_1', X_2'$ ヲ得タリトセバ、是等ノ二雙ノ點ハ所要ノ點ナルベシ。

平面上ニ於ケル此問題ノ初等解法如何？ (T. K.)

石黒藤右衛門ノ贈位

大正六年十一月十七日陸軍大演習ニ就キ滋賀縣ニ行幸アリ。其際他ノ勤王諸家碩學鴻儒等ト共ニ、石黒藤右衛門ニ贈從五位ノ恩典ヲ賜ハル。石黒氏諱ハ信由、藤右衛門ト稱ス。山路主徴及ビ藤田定資ニ就キテ學ベル田中高寛ノ高弟ナリ。越中國射水郡高木村ノ人ニシテ、文化十年 1813 算學鈎致及ビ渡海標のヲ著ハシ、其他寫本ニシテ同人ノ撰セルモノ多シ。和算學ニ長

ジ、測量製圖ニ精シク、其研鑽天文曆學ニ及ブ。寛政天保 1780-1843 ノ頃屢々檢地測量地圖製作ニ從事シ、用水設計山野開墾ニ盡力セリ。

因ニ記ス。余大正六年 1917 十一月發行ニカカル東京物理學校雜誌第 312 號ニ於テ「方陣問題ノ調査資料ニ就テ」ト題スル論文ヲ公ケニセシガ、文中蟹瀬厚義ノ事ニ及ビ、其ノ何人ナルヲ知ラズト記セリ。然ルニ其後此ノ拙論ヲ讀マレテ余ニ其人トナリヲ明示セラレタル余ノ友人アリ。之ヲ金澤第一中學校教諭野島乙松氏トス。同氏ノ通信ニヨレバ蟹瀬厚義ハ石黒藤右衛門ノ高弟ニシテ越中國礪波郡内島村ニ住シ、五十嵐小豐治(或ハ孫作)厚義(或ハ篤好)ノコトノ由ニテ野島氏令聞ハ厚義ノ孫ナリト。サレバ拙文中ノ四方陣變化ナル書ハ石黒ノ著ニシテ蟹瀬ノ寫シ取リシモノナリト推セラル。蟹瀬ノ遺書亦多クシテ金澤市内ニ散在セリト聞ク。(T. H.)

中 等 學 校 學 生 用 ノ 數 學 雜 誌

此種ノ雜誌ノ興廢ハ恒ナシトモ云ハルルガ如キ狀態ナルガ、松岡文太郎氏ノ主管セル「數學雜誌」(明治三十五年創刊)ト長澤龜之助氏ノ主管セル「XY」(明治三十九年創刊)トハ現時ニ於ケル有力ナルモノナルベシ。然ルニ所謂數學ノ考ヘ方ノ主唱者トシテ、又之ヲ表題トセル參考書ノ著作者トシテ、更ニ又其ノ宣傳ノ爲ニ日土講習會ノ講師トシテ、中等學校學生ノ間ニ好評アル藤森良藏氏ハ大正六年 1917 九月ヨリ毎月一回「受験豫備考ヘ方」ナル雜誌ヲ發刊セラレ歡迎ヲ受クルモノナルヘシ。

京 都 及 ビ 仙 臺 ノ 理 科 大 數 學 學 科 教 師 ノ 擔 任 課 目

本學年 1917-1918 ニ於ケル兩大學ノ擔任次ノ如シ。亞刺比亞數字ハ每週ノ時間數ナリ。

(京都) 河合教授：函數論總論(一年間 3)、同各論(一年間 3)、數學解析大要(九月ヨリ十二月マデ 3、一月ヨリ六月マデ 2)、和田助教授：微分、積分、微分方程式(一年間 5)、射影幾何學總論(一年間 2)、西内助教授：射影幾何學特論(一月ヨリ六月マデ 3)、園助教授：代數學(一年間 2)、數論(一年間 3)、松本講師：立體解析幾何學(一年間 2)、同演習(一年間 3)、杉谷講師：微分幾何學(一年間 2)、安田講師：微分、積分、微分方程式演習(一年間 2)。

(仙臺) 林教授：代數學(2)、微分幾何學(2)、同演習(隔週 2)、特選題目(算術的函數及ビ素數ノ配列)(2)、數學研究(3)、藤原教授：微分、積分(4)、同演習(8)、實變數ノ函數論(2)同演習(隔週 2)、實用數學(2)、窪田教授：座標幾何學(九月ヨリ十二月マデ 3、一月ヨリ六月マデ 2)、同演習(3)、複素變數ノ函數論(3)、同演習(3)、掛谷助教授：代數解析(2)、特選題目(2)、柴山講師：微分方程式(3)。
[期間ハ明記セルモノノ外皆一年間ナリ]

二 三 雜 誌 中 ノ 注 目 ス ベ キ 論 說 記 事

東京物理學校雜誌、大正 6 年 11 月號

方陣問題調査資料ニ就テ

理學博士 林 鶴 一 氏

哲學雜誌、大正 6 年 10,11 月號

エルンスト、マツハの思想

理學士 阿 部 良 夫 氏

スピノザ哲學に於ける認識問題

文學士 出 隆 氏

保險雜誌、大正 6 年 1 月號——11 月號

森村、一年ノ小數期間毎ニ支拂ハルル生命保險料ニ就テ、納質、支拂保險金ノ實損額ト保險料ノ理論的解剖、龜田、責任準備監査ノ一方法、森村、 $N_x^{(m)}$ 表作成方法ノ補遺、興石、變數年金計算法、角尾、官營簡易生命保險ノ豫定解約率並加入見込人員表ヲ論ズ、角尾、生命保險業ノ解約支拂戻金ニ就テ、門脇、附加保險料附加方法ニ就テ、角尾、我國ニ於ケル生命保險ノ保

險料=就テ、門脇、保險料積立金群團計算公式、角尾、生命保險ノ拂濟保險金額並解約返戻金算出方法=就テ、森村、 $N_x^{(m)}$ 及 $R_x^{(m)}$ ノ算式=就テ、

(保險界=精密ナル數理的研究ノ近時特=目立チテ殖エタルハ實=慶賀スベキコトナリ、空疎茫漠タル保險思想ハ排スベシ)。

The Monist, Vol. 27, 1917.

A. E. Heath, Hermann Grassmann (1809-1877). A. E. Heath, The neglect of the work of Grassmann. A. E. Heath, The geometrical analysis of Grassmann and its connection with Leibniz's characteristic. D. M. Wrinch, Bernardo Bolzano (1781-1848). G. Frege, Class, function, concept, relation. P. Carus, Leibniz and Locke. Ph. E. B. Jourdain, Existents and entities. J. M. Child, The manuscripts of Leibniz on his discovery of the differential calculus. Part 2. D. E. Smith, Notes on De Morgan's Budge. of paradoxes. K. I. Gerhardt, Leibniz in London (Translated with critical notes by J. M. Child, from an article by Gerhardt in the Berliner Berichte, 1891, pp. 157-165).

C M Hebbert 氏並= W. S. Baer 氏ノ論文

本號所載ノ C. M. Hebbert 氏ノ論文ハ同氏ガ米國 Illinois 大學ニ於テ Doctor ノ資格ヲ得タル受験論文ノ一部ヲ爲スモノナルガ、讀者ハ之ヲ以テ米國 Doctor ノ程度ヲ知ルノ一資料トナスヲ得ン。又前卷ニ其ノ論文ヲ發表セル獨逸人(現ニ西比利亞ニ捕虜トシテ滞在セル) Dr. W. S. Baer ハ同卷第 234 頁ニ記載セルモノノ外 Math. Ann. Bd 76 (1915) S. 284 = F. Bernstein 氏トノ共著トシテ Ein Axiomensystem der Methode der kleinsten Quadrate ナル論文アリ。(T. H.)

諸 學 者 ノ 消 息

英吉利けむぶりっち大學 St. John's College ノ fellow ニシテ且 University Lecturer タリシベざんと Dr. W. H. Besant ハ 1917 年 6 月 2 日八十八歳ニテ逝去セリ。

加奈陀とろんと大學びいち Dr. Samuel Beatty ハ同大學數學助教授トナレリ。

伊太利すとらぜり Dr. V. Strazzeri ハばれるも大學ニ於ケル解析幾何學及 射影幾何學ノ教師トナレリ。

佛蘭西ばり大學ばんるぐえ Dr. P. Painlevé ハ既報ノ如ク首相兼陸相タリシモ辭職シタリ。

北米合衆國しかご大學ノらん A. C. Lunn ハ同大學ノ Associate Professor トナレリ。

北米合衆國わしんとん大學ノろく W. H. Rover ハ同大學教授トナレリ。

東北帝國大學理科大学講師理學士寺澤寛一氏ハ同大學數學科及物理學科ニ於テハ力學ヲ、應用化學科ニ於テハ數學力學ヲ講演セラレツ、アルガ、大正六年十二月十二日論文提出、東北帝國大學理科大学教授會ノ審査ニヨリテ理學博士ノ學位ヲ授ケラレタリ。論文ハ「一方へ無限ニ擴ガレル固體ノ與ヘラレタル境界條件ノ下ニ於ケル彈性的平衡ノ論竝ニ其ノ應用」(英文)ナリ。又同氏ハ大正六年 1917 十二月二十八日東北帝國大學工學專門部教授ニ任ゼラレ同七年 1918 一月十二日理科大学講師囑托ヲ解カレタルガ、其ノ擔任講義ハ上記ノモノト異ナラザル筈ナリ。

京都帝國大學理科大学助教授和田健雄氏ハ大正六年 1917 十一月二十一日數學解析研究ノ爲滿二箇年間米國へ留學ヲ命ゼラレタリ。

京都帝國大學理科大学助教授西内貞吉氏ハ大正六年 1917 十二月五日歸朝セラレタルガ、別項記載ノ如ク射影幾何學特論ヲ講ゼラルモノナラン。

A Serious Misprint

took place in the paper of Mr. Nilos Sakellariou, Vol. 11, pp. 75-83. The lines between the 9th line on p. 82 and the 11th line on p. 83 must be inserted next to the 6th line on p. 81, and the numbers of articles 6 and 7 must be interchanged.

In addition to this the following errata should be noticed in his papers :

p. 76, l. 4.	Read	$-\frac{1}{\rho} \sin \varphi \frac{\partial v}{\partial \varphi}$	for	$-\frac{1}{\rho} \sin \varphi \frac{\partial v}{\partial \rho}$
p. 76, l. 6.	„	$+\frac{1}{\rho} \cos \varphi \frac{\partial v}{\partial \varphi}$	„	$+\frac{1}{\rho} \cos \varphi \frac{\partial v}{\partial \rho}$
p. 76, l. 11.	„	unité	„	unite
p. 79, l. 10.	„	$+\frac{1}{\rho} \cos \varphi \frac{\partial \Phi}{\partial \varphi}$	„	$+\frac{1}{\rho} \cos \varphi \frac{\partial \Phi}{\partial \rho}$
p. 79, ls. 9, 10.	Give the number (9) to these two formulas.			
p. 85, l. 10.	Read	ΔZ	for	Δz .
p. 85, ls. 24, 25.	Put	Z	for	z .
p. 86, l. 24.	Give the number (5'') to the formula.			
p. 88, l. 11.	Read	von r	for	von z .
p. 89, l. 21.	Read	(2)	for	(3).

Moreover, the name of the author must be Sakellariou, but not Sakellariou.

Editor.

On the Problem of the Calculus of Variations in n Dimensions,

by

PAUL R. RIDER, Saint Louis, U.S.A.

Introduction.

For the study of the simplest problem of the calculus of variations in n dimensions two forms of integral at once suggest themselves, namely,

$$\int_{x_0}^{x_1} f(x, y_1, \dots, y_{n-1}, y'_1, \dots, y'_{n-1}) dx, \quad y'_i = \frac{dy_i}{dx},$$

$$\int_{t_0}^{t_1} F(x_1, \dots, x_n, x'_1, \dots, x'_n) dt, \quad x'_i = \frac{dx_i}{dt}.$$

Of more interest however is the integral

$$(1) \quad I = \int_{t_0}^{t_1} f(x_1, \dots, x_n, \tau_1, \dots, \tau_{n-1}) \sqrt{x_1'^2 + \dots + x_n'^2} dt \quad (1).$$

The x 's are functions of t defined by the curve

$$(C) \quad x_i = \phi_i(t), \quad i = 1, \dots, n.$$

The angle τ_{n-1} is the angle made by the positive tangent to the curve C with its projection in the $(n-1)$ -space (x_1, \dots, x_{n-1}) , τ_{n-2} is the angle made by this projection with its own projection in the $(n-2)$ -space (x_1, \dots, x_{n-2}) , etc. Analytically the τ 's are defined by the equations

$$(2) \quad \tau_i = \arctan \frac{x'_{i+1}}{\sqrt{x_1'^2 + \dots + x_i'^2}}, \quad i = 1, \dots, n-1,$$

(1) Bliss has given the theory for an integral of the form $\int f(x, y, \tau) \sqrt{x'^2 + y'^2} dt$, $\tau = \arctan \frac{y'}{x'}$, in *A generalization of the notion of angle*, Transactions of the American Mathematical Society, vol. 7 (1906), pp. 184-199; and *A new form of the simplest problem of the calculus of variations*, *ibid.*, vol. 8 (1907), pp. 405-414.

In a paper entitled *The space problem of the calculus of variations in terms of angle*, American Journal of Mathematics (July, 1917), I have treated the integral

$$\int f(x, y, z, \tau, \sigma) \sqrt{x'^2 + y'^2 + z'^2} dt, \quad \tau = \arctan \frac{y'}{x'}, \quad \sigma = \arctan \frac{z'}{\sqrt{x'^2 + y'^2}}.$$

from which it is seen that

$$(3) \quad \frac{x'_i}{\sqrt{x_1'^2 + \dots + x_n'^2}} = \sin \tau_{i-1} \cos \tau_i \dots \cos \tau_{n-1}, \quad i=1, \dots, n-1$$

(if we make the convention $\sin \tau_0 = 1$).

In section I the Euler equations for the integral (1) are derived, and the form of the solutions of these equations is shown; the transversality and corner conditions are stated in section II; in section III the forms of the Hilbert invariant integral and the Weierstrass e -function are given, and certain necessary conditions, including the condition of Weierstrass, are obtained.

I. The Euler equations.

Let us then proceed to the consideration of the integral (1). We assume that the integrand function $f(x_1, \dots, x_n, \tau_1, \dots, \tau_{n-1})$ is of class $C^{IV(1)}$ with respect to each of its $2n-1$ arguments in a region $R(x_1, \dots, x_n, \tau_1, \dots, \tau_{n-1})$ and shall throughout the discussion confine our considerations to this region. The curves employed will be represented in the parametric form (C) , where ϕ_1, \dots, ϕ_n are of class C^{IV} and define values $(x_1, \dots, x_n, \tau_1, \dots, \tau_{n-1})$ interior to the region R for all values of t in the interval $t_0 \leq t \leq t_1$. Moreover we assume that x'_1, \dots, x'_n do not all vanish simultaneously.

Suppose that all curves of the family

$$(C_a) \quad x_i = \phi_i(t, a), \quad i=1, \dots, n,$$

where ϕ_1, \dots, ϕ_n are of class C^{IV} in the region

$$t_0 \leq t \leq t_1, \quad |a| \leq c > 0,$$

pass through the two fixed points $P_0(t_0)$ and $P_1(t_1)$, the end points of the curve C , which is supposed to furnish a minimum for the integral I . Furthermore, let the family C_a define C for $a=0$. Then if the integral I is taken along a curve of the family, its value becomes a function of a , and since C minimizes I , $\frac{dI}{da}$ must vanish for $a=0$.

We find that

(1) The terms and notations used are those of Bolza, *Vorlesungen über Variationsrechnung*.

$$(4) \quad \frac{dI}{da} = \int_{t_0}^{t_1} \left(f_{x_1} \frac{\partial x_1}{\partial a} + \dots + f_{x_n} \frac{\partial x_n}{\partial a} + f_{\tau_1} \frac{\partial \tau_1}{\partial a} + \dots + f_{\tau_{n-1}} \frac{\partial \tau_{n-1}}{\partial a} \right. \\ \left. + f \frac{x_1' \frac{\partial x_1'}{\partial a} + \dots + x_n' \frac{\partial x_n'}{\partial a}}{x_1'^2 + \dots + x_n'^2} \right) \sqrt{x_1'^2 + \dots + x_n'^2} dt.$$

From (2),

$$(5) \quad \frac{\partial \tau_i}{\partial a} = \frac{(x_1'^2 + \dots + x_i'^2) \frac{\partial x'_{i+1}}{\partial a} - x'_{i+1} \left(x_1' \frac{\partial x'_1}{\partial a} + \dots + x_i' \frac{\partial x'_i}{\partial a} \right)}{(x_1'^2 + \dots + x_{i+1}'^2) \sqrt{x_1'^2 + \dots + x_i'^2}}. \\ i=1, \dots, n-1.$$

Substituting (5) in (4), and replacing t by s , the length of arc, we find upon using (2) or (3) that

$$\frac{dI}{da} = \int_{s_0}^{s_1} \sum_{i=1}^n \left[f_{x_i} \frac{\partial x_i}{\partial a} + f \sin \tau_{i-1} \cos \tau_i \dots \cos \tau_{n-1} \frac{\partial x'_i}{\partial a} \right. \\ \left. + \frac{f \tau_i}{\cos \tau_{i+1} \dots \cos \tau_{n-1}} \left\{ \cos \tau_i \frac{\partial x'_{i+1}}{\partial a} - \sin \tau_i \left(\cos \tau_1 \dots \cos \tau_{i-1} \frac{\partial x'_1}{\partial a} \right. \right. \right. \\ \left. \left. \left. + \sin \tau_1 \cos \tau_2 \dots \cos \tau_{i-1} \frac{\partial x'_2}{\partial a} + \dots + \sin \tau_{i-1} \frac{\partial x'_i}{\partial a} \right) \right\} \right] ds.$$

The accents now denote differentiation with respect to s . (It is understood that $f_{\tau_n} = 0$.)

It follows in the usual way that the *Euler equations* are

$$(6) \quad f_{x_i} - \frac{dp_i}{ds} = 0, \quad i=1, \dots, n,$$

where

$$(7) \quad p_i = f \sin \tau_{i-1} \cos \tau_i \dots \cos \tau_{n-1} + f_{\tau_{i-1}} \frac{\cos \tau_{i-1}}{\cos \tau_{i-1} \dots \cos \tau_{n-1}} \\ - \sum_{j=i}^{n-1} f_{\tau_j} \frac{\sin \tau_j \sin \tau_{i-1} \cos \tau_i \dots \cos \tau_{j-1}}{\cos \tau_{j+1} \dots \cos \tau_{n-1}}.$$

The n equations (6) are not independent. For, designating them by $P_1=0, \dots, P_n=0$, we can easily show, after carrying out the differentiation, that there exists the relation

$$(8) \quad \sum_{i=1}^n P_i \sin \tau_{i-1} \cos \tau_i \dots \cos \tau_{n-1} = 0.$$

When the differentiation has been performed and the terms properly grouped, equations (6) have the form

$$(9) \quad a_{i1} \frac{d\tau_1}{ds} + \dots + a_{i, n-1} \frac{d\tau_{n-1}}{ds} + b_i = 0, \quad i=1, \dots, n.$$

These n equations in the $n-1$ unknowns $\frac{d\tau_1}{ds}, \dots, \frac{d\tau_{n-1}}{ds}$ are consistent because of the linear relation (8); therefore if at least one of the $(n-1)$ -rowed determinants of the matrix

$$\begin{vmatrix} a_{11} & \dots & a_{1, n-1} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{n, n-1} \end{vmatrix}$$

is different from zero, we can solve them and obtain

$$(10) \quad \frac{d\tau_i}{ds} = A_i(x_1, \dots, x_n, \tau_1, \dots, \tau_{n-1}), \quad i=1, \dots, n-1.$$

From (3) we have

$$(11) \quad \frac{dx_i}{ds} = \sin \tau_{i-1} \cos \tau_i \dots \cos \tau_{n-1}, \quad i=1, \dots, n.$$

It follows from existence theorems for differential equations that in the region R , through the point (x_1^0, \dots, x_n^0) and in the direction defined by $(\tau_1^0, \dots, \tau_{n-1}^0)$, we can draw one and but one extremal. The equations of these extremals are obtained by integrating (10) and (11). They have the form

$$x_i = \phi_i(s, x_1^0, \dots, x_n^0, \tau_1^0, \dots, \tau_{n-1}^0), \quad i=1, \dots, n.$$

Furthermore, these initial conditions are satisfied:

$$\begin{aligned} \phi_i(0, x_1^0, \dots, x_n^0, \tau_1^0, \dots, \tau_{n-1}^0) &= x_i^0, \\ \frac{d}{ds} \phi_i(0, x_1^0, \dots, x_n^0, \tau_1^0, \dots, \tau_{n-1}^0) &= \sin \tau_{i-1}^0 \cos \tau_i^0 \dots \cos \tau_{n-1}^0. \end{aligned}$$

II. The transversality and corner conditions.

The Euler equations of section I were derived with the assumption that both end points P_0, P_1 of the curve C are fixed. It can be shown that if either end point is allowed to vary along a given curve T , then at this point the relation

$$\sum_{i=1}^n (f \sin \tau_{i-1} \cos \tau_i \dots \cos \tau_{n-1} + f_{\tau_{i-1}} \frac{\cos \tau_{i-1}}{\cos \tau_i \dots \cos \tau_{n-1}})$$

$$-\sum_{j=l}^{n-1} f_{\tau_j} \frac{\sin \tau_j \sin \tau_{j-1} \cos \tau_l \dots \cos \tau_{l-1}}{\cos \tau_{j+1} \dots \cos \tau_{n-1}} \Big) \sin \bar{\tau}_{l-1} \cos \bar{\tau}_l \dots \cos \bar{\tau}_{n-1} = 0$$

must hold, where the angles $\bar{\tau}_1, \dots, \bar{\tau}_{n-1}$ define direction on I .

At a corner point $P_2(t_2)$ on a broken extremal $P_0 P_2 P_1$ the equations

$$\begin{aligned} & f \sin \tau_{l-1} \cos \tau_l \dots \cos \tau_{n-1} + f_{\tau_{l-1}} \frac{\cos \tau_{l-1}}{\cos \tau_l \dots \cos \tau_{n-1}} \\ & - \sum_{j=l}^{n-1} f_{\tau_j} \frac{\sin \tau_j \sin \tau_{l-1} \cos \tau_l \dots \cos \tau_{j-1}}{\cos \tau_{j+1} \dots \cos \tau_{n-1}} \\ & = \bar{f} \sin \bar{\tau}_{l-1} \cos \bar{\tau}_l \dots \cos \bar{\tau}_{n-1} + \bar{f}_{\tau_{l-1}} \frac{\cos \bar{\tau}_{l-1}}{\cos \bar{\tau}_l \dots \cos \bar{\tau}_{n-1}} \\ & - \sum_{j=l}^{n-1} \bar{f}_{\tau_j} \frac{\sin \bar{\tau}_j \sin \bar{\tau}_{l-1} \cos \bar{\tau}_l \dots \cos \bar{\tau}_{j-1}}{\cos \bar{\tau}_{j+1} \dots \cos \bar{\tau}_{n-1}}, \quad i=1, \dots, n, \end{aligned}$$

must be satisfied⁽¹⁾, where $\tau_1, \dots, \tau_{n-1}$ and $\bar{\tau}_1, \dots, \bar{\tau}_{n-1}$ define the directions of the arcs $P_0 P_2$ and $P_2 P_1$ respectively at P_2 . A bar over a function signifies that its arguments are $x_1, \dots, x_n, \bar{\tau}_1, \dots, \bar{\tau}_{n-1}$.

III. Certain necessary conditions.

Hilbert's invariant integral, taken along a curve \bar{C} of parameter a in a field of extremals, is discovered by the usual methods to be

$$I^* = \int_{i=1}^n p_i \sin \bar{\tau}_{l-1} \cos \bar{\tau}_l \dots \cos \bar{\tau}_{n-1} \sqrt{\phi_1'^2 + \dots + \phi_n'^2} da,$$

accents denoting differentiation with respect to a . The Weierstrass e -function is the function

$$e(x_1, \dots, x_n; \tau_1, \dots, \tau_{n-1}; \bar{\tau}_1, \dots, \bar{\tau}_{n-1}) = \bar{f} - \sum_{i=1}^n P_i \sin \bar{\tau}_{l-1} \cos \bar{\tau}_l \dots \cos \bar{\tau}_{n-1}$$

occurring in the integrand of the difference $I - I^*$. Weierstrass's necessary condition follows at once:

The condition

$$e(x_1, \dots, x_n; \tau_1, \dots, \tau_{n-1}; \bar{\tau}_1, \dots, \bar{\tau}_{n-1}) \geq 0$$

must be satisfied at every point of a minimizing curve and for every direction $\bar{\tau}_1, \dots, \bar{\tau}_{n-1}$.

(1) For the special cases $n=2$ and $n=3$ I have already given these conditions. See *A note on discontinuous solutions in the calculus of variations*, Bulletin of the American Mathematical Society, vol. 23, no. 5 (Feb, 1917), pp. 237-240.

To deduce further necessary conditions, let us expand, by Taylor's series with a remainder, each term of e considered as a function of $\bar{\tau}_1, \dots, \bar{\tau}_{n-1}$. We find that

$$(12) \quad \bar{f} = f + \sum_{i=1}^{n-1} (\bar{\tau}_i - \tau_i) f_{\tau_i} + \frac{1}{2} \sum_{i=1, j=1}^{n-1} (\bar{\tau}_i - \tau_i)(\bar{\tau}_j - \tau_j) f_{\tau_i \tau_j} + h_1,$$

where the remainder h_1 is homogeneous of degree three in $\bar{\tau}_1 - \tau_1, \dots, \bar{\tau}_{n-1} - \tau_{n-1}$ and contains third partial derivatives of f with mean value arguments. For the other terms of the e -function we find that

$$(13) \quad \sum_{i=1}^n p_i \sin \bar{\tau}_{i-1} \cos \bar{\tau}_i \dots \cos \bar{\tau}_{n-1} = f + \sum_{i=1}^{n-1} (\bar{\tau}_i - \tau_i) f_{\tau_i} \\ - \frac{1}{2} \sum_{i=1}^{n-1} (\bar{\tau}_i - \tau_i)^2 (f \cos^2 \tau_{i+1} \dots \cos^2 \tau_{n-1} - \sum_{j=i+1}^{n-1} f_{\tau_j} \sin \tau_j \cos \tau_j \cos^2 \tau_{i+1} \dots \cos^2 \tau_{j-1}) \\ - \sum_{i=1, j=i+1}^{n-1} (\bar{\tau}_i - \tau_i)(\bar{\tau}_j - \tau_j) f_{\tau_i} \tan \tau_j + h_2,$$

where h_2 has the properties possessed by h_1 . If we subtract (13) from (12) it is seen that

$$e = \frac{1}{2} \sum_{i=1}^{n-1} (\bar{\tau}_i - \tau_i)^2 (f \cos^2 \tau_{i+1} \dots \cos^2 \tau_{n-1} \\ - \sum_{j=i+1}^{n-1} f_{\tau_j} \sin \tau_j \cos \tau_j \cos^2 \tau_{i+1} \dots \cos^2 \tau_{j-1} + f_{\tau_i \tau_i}) \\ + \sum_{i=1, j=i+1}^{n-1} (\bar{\tau}_i - \tau_i)(\bar{\tau}_j - \tau_j) (f_{\tau_i} \tan \tau_j + f_{\tau_i \tau_j}) + h.$$

The quantity h , being equal to $h_1 - h_2$, is homogeneous of degree three in $\bar{\tau}_1 - \tau_1, \dots, \bar{\tau}_{n-1} - \tau_{n-1}$ and contains third partial derivatives of f with mean value arguments. As we have assumed that f is of class C^{IV} with respect to all of its arguments, h is finite and vanishes as all of the quantities $\bar{\tau}_1 - \tau_1, \dots, \bar{\tau}_{n-1} - \tau_{n-1}$ approach zero. (It is assumed that $f \cos^2 \tau_{i+1} \dots \cos^2 \tau_{n-1} = f$ when $i = n-1$.)

Let us define a quadratic form

$$Q = \sum_{i=1}^{n-1} a_{ii} \hat{\xi}_i^2 + 2 \sum_{i=1, j=i+1}^{n-1} a_{ij} \hat{\xi}_i \hat{\xi}_j$$

in the real variables $\hat{\xi}_1, \dots, \hat{\xi}_{n-1}$, by setting

$$a_{ii} = f \cos^2 \tau_{i+1} \dots \cos^2 \tau_{n-1} - \sum_{j=i+1}^{n-1} f_{\tau_j} \sin \tau_j \cos \tau_j \cos^2 \tau_{i+1} \cos^2 \tau_{j-1} + f_{\tau_i \tau_i}, \\ a_{ij} = f_{\tau_i} \tan \tau_j + f_{\tau_i \tau_j}.$$

Then Q must be greater than or equal to zero at all points of a minimizing curve and for all values of $\hat{\xi}_1, \dots, \hat{\xi}_{n-1}$.

For if $\hat{\xi}_1, \dots, \hat{\xi}_{n-1}$ were a set of values making Q negative at some point of the minimizing curve, we could choose $\bar{\tau}_1 - \tau_1 = \varepsilon \hat{\xi}_1, \dots, \bar{\tau}_{n-1} - \tau_{n-1} = \varepsilon \hat{\xi}_{n-1}$, and for sufficiently small values of ε the e -function would be negative. But we have seen in the first part of this section that for a minimum e cannot be negative.

Furthermore, from the theory of quadratic forms, it follows that a_{ii} must not be negative, that is,

$$f \cos^2 \tau_{i+1} \dots \cos^2 \tau_{n-1} - \sum_{j=i+1}^{n-1} f_{\tau_j} \sin \tau_j \cos \tau_j \cos^2 \tau_{i+1} \dots \cos^2 \tau_{j-1} + f_{\tau_i \tau_i} \geq 0, \\ i=1, \dots, n-1.$$

Washington University,
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A Geometrical Study of the Mechanics of a Particle⁽¹⁾,

by

KINNOSUKE OGURA, Ôsaka.

The line geometry plays the fundamental rôle in the mechanics of rigid bodies, and the theory of infinitesimal deformation of curved plates leads us to the beautiful researches of Weingarten, Ribaucour, Guichard, Koenigs, etc. in the differential geometry of congruence.

The primary object of this paper is to deal with the mechanics of a particle from the standpoint of line geometry. Since the totality of all the acting lines of force in a general field constitutes a line complex, I have used the coordinates of force. This system of coordinates is not only of advantage in the treatment of the distribution of the acting lines of force, but it is valuable in the theory of Appell transformations.

The Appell transformation, i.e. the collineation combined with a certain time transformation is of peculiar importance in the dynamics of a particle; and hence it is desirable to find the force under which the collineation can be produced. The study of this force (the *W-force*) is the secondary object of this paper.

Since all the acting lines of *W-force* form a tetrahedral complex, there are some interesting figures and transformations connected with this force. In Part III, I have dealt with such geometrical transformations, especially *the contact transformation in which a straight line corresponds to a certain Kummer surface.*

In later papers⁽²⁾ I expect to make a general study of the point-line connexion, such as has been appeared in the field of *W-force*.

⁽¹⁾ Read before the Tôkyô Mathematico-Physical Society, September 15, 1917, under a different title.

⁽²⁾ See my paper "Theory of the point-line connexion in space" which will appear in this Journal in the near future.

PART I. MOTION IN A PLANE.

General force.

1. Consider the field of positional force defined by the equations of motion

$$\frac{d^2x}{dt^2} = X(x, y), \quad \frac{d^2y}{dt^2} = Y(x, y).$$

Then the equation to the line of action of the force (the *force-line*, as it is now called) at the point (x, y) is

$$(1) \quad \eta X - \xi Y + (xY - yX) = 0^{(1)},$$

(ξ, η) standing for the current point coordinates.

If we introduce the *coordinates of force* (p_1, p_2, p_3) :

$$(2) \quad p_1 = -Y, \quad p_2 = X, \quad p_3 = xY - yX^{(2)},$$

equation (1) becomes

$$p_1\xi + p_2\eta + p_3 = 0;$$

so that (p_1, p_2, p_3) may be considered as homogeneous line coordinates (in the sense of Plücker) of the force-line at (x, y) .

Again if we use the homogeneous point coordinates such that

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3},$$

we have from (2) the relation of incidence:

$$(3) \quad p_1x_1 + p_2x_2 + p_3x_3 = 0.$$

When the point (x_1, x_2, x_3) describes an orbit, the corresponding force-line envelopes a curve which may be called the *force-envelope* corresponding to the orbit.

2. If a point (x_1, x_2, x_3) be given, there is one force-line (p_1, p_2, p_3) acting at that point; conversely if a line be given, there are some points where each of the force-lines coincides with the given line, unless all the acting lines in the field envelope a curve

$$f(p_1, p_2, p_3) = 0;$$

(1) We assume that the force-line is not orientated.

(2) p_3 denotes the moment of force about the origin of the coordinates.

that is, unless there exists a relation among the force-components and the moment of force about a point⁽¹⁾. Hence a given field determines a certain correspondence between points and lines.

It is now desirable to recall Clebsch's theory of connex. Let us take a principal coincidence of the first order and the m^{th} class defined by

$$(4) \quad \begin{cases} p_1 f_1 + p_2 f_2 + p_3 f_3 = 0, \\ p_1 x_1 + p_2 x_2 + p_3 x_3 = 0, \end{cases}$$

where f_1, f_2, f_3 are homogeneous polynomials of the m^{th} degree with respect to x_1, x_2, x_3 . Then we find

$$\rho x_3^{m+1} \cdot p_1 = f_2 x_3 - x_2 f_3, \quad \rho x_3^{m+1} \cdot p_2 = f_3 x_1 - x_3 f_1, \quad \rho x_3^{m+1} \cdot p_3 = f_1 x_2 - x_1 f_2,$$

ρ being an arbitrary function of x_1, x_2, x_3 . Therefore we can construct infinitely many fields of force by the given principal coincidence of the first order.

A curve drawn from any origin, so that at every point on it its tangent is the force-line at that point, is called a *line of force*. Therefore the lines of force consist of the ∞^1 curves determined by

$$\frac{dy}{dx} = \frac{Y}{X};$$

and in our case these are nothing but the curves of principal coincidence (4):

$$f_1 \cdot (x_3 dx_2 - x_2 dx_3) + f_2 \cdot (x_1 dx_3 - x_3 dx_1) + f_3 \cdot (x_2 dx_1 - x_1 dx_2) = 0.$$

Hence the lines of force in the field defined by (4) are panalgebraic curves of the first degree and the m^{th} rank, which have been treated by Clebsch⁽²⁾, Darboux⁽³⁾ and Prof. Loria⁽⁴⁾.

3. Prof. Appell proved that by the collineation

$$x' = \frac{ax + by + c}{a''x + b''y + c''}, \quad y' = \frac{a'x + b'y + c'}{a''x + b''y + c''}$$

combined with the time transformation

$$k dt' = \frac{dt}{(a''x + b''y + c'')^2}, \quad (k \text{ being any const.})$$

(1) The central force belongs to this exceptional case. In what follows we will not consider this exceptional case unless the contrary is stated.

(2) Clebsch-Lindemann, Vorlesungen über Geometrie, I. 2 (1876), p. 962.

(3) Darboux, Bull. Sc. math., (2) 2 (1878). See Jordan, Cours d'analyse, 3 (2. éd., 1896), p. 28.

(4) Loria, Spezielle ebene Kurven, 1. Aufl. (1902), p. 724.

the equations of motion

$$\frac{d^2 x}{dt^2} = X, \quad \frac{d^2 y}{dt^2} = Y$$

become

$$\frac{d^2 x'}{dt'^2} = X', \quad \frac{d^2 y'}{dt'^2} = Y',$$

where

$$(5) \quad \begin{cases} X' = k^2(a''x + b''y + c'')^2 \{B'X - A'Y + C'(xY - yX)\}, \\ Y' = k^2(a''x + b''y + c'')^2 \{-BX + AY - C(xY - yX)\}^{(1)}; \end{cases}$$

and therefore

$$(6) \quad x'Y' - y'X' = k^2(a''x + b''y + c'')^2 \{B''X - A''Y + C''(xY - yX)\},$$

A, B, \dots standing for the algebraic complements of a, b, \dots in the determinant

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

These equations (5), (6) may be written *symmetrically* by using the coordinates of force: For, it is well known that by the collineation

$$\lambda x'_1 = ax_1 + bx_2 + cx_3,$$

$$\lambda x'_2 = a'x_1 + b'x_2 + c'x_3,$$

$$\lambda x'_3 = a''x_1 + b''x_2 + c''x_3,$$

the homogeneous line coordinates are transformed into

$$(7) \quad \begin{cases} \rho p'_1 = Ap_1 + Bp_2 + Cp_3, \\ \rho p'_2 = A'p_1 + B'p_2 + C'p_3, \\ \rho p'_3 = A''p_1 + B''p_2 + C''p_3. \end{cases}$$

Hence if we put

$$k dt' = \frac{x_3^2 dt}{(a''x_1 + b''x_2 + c''x_3)^2}$$

and

$$\rho = \frac{x_3^2}{k^2(a''x_1 + b''x_2 + c''x_3)^2},$$

equations (7) are nothing but (5) and (6). Thus *the reason why the*

(¹) Appell, Amer. J. of math., 12 (1890), p. 103.

transformation of force takes the form (5), up to the proportional factor, may be easily explained.

Similarly if we introduce the coordinates of velocity

$$(8) \quad q_1 = -\frac{dy}{dt}, \quad q_2 = \frac{dx}{dt}, \quad q_3 = x \frac{dy}{dt} - y \frac{dx}{dt},$$

by the Appell transformation these become

$$(9) \quad \begin{cases} \frac{1}{k} q'_1 = A q_1 + B q_2 + C q_3, \\ \frac{1}{k} q'_2 = A' q_1 + B' q_2 + C' q_3, \\ \frac{1}{k} q'_3 = A'' q_1 + B'' q_2 + C'' q_3. \end{cases}$$

4. We can form various geometries from the standpoint of the dynamics of a particle.

Prof. Kasner proved that collineation is the only point transformation which transforms all the orbits under one (positional) force into those under another (positional) force; conversely any collineation

$$x' = \frac{ax + by + c}{a''x + b''y + c''}, \quad y' = \frac{a'x + b'y + c'}{a''x + b''y + c''}$$

transforms the orbits under any given force X, Y into the orbits under the force X', Y' , where

$$(5) \quad \begin{cases} X' = k^2(a''x + b''y + c'')^2 \{B'X - A'Y + C'(xY - yX)\}, \\ Y' = k^2(a''x + b''y + c'')^2 \{-BX + AY - C(xY - yX)\}, \end{cases}$$

k being an arbitrary constant⁽¹⁾.

Next, consider the collineation which transforms all the orbits under one parallel force into those under another parallel force. By this transformation

$$Y = \lambda X \quad (\lambda \text{ being any constant})$$

becomes

$$Y' = \lambda' X',$$

where λ' is a constant depending upon λ . Hence it follows from (5) that

$$\lambda' = \frac{-B + A\lambda - C(x\lambda - y)}{B' - A'\lambda - C'(x\lambda - y)}$$

(1) Kasner, Differential-geometric aspects of dynamics (The Princeton colloquium lectures on mathematics, 1913).

or

$$(\lambda' C' + C)(y - \lambda x) = \lambda'(B' - \lambda A') + (B - \lambda A)$$

must hold good for any value of x and y ; so that

$$\lambda' C' + C = 0, \quad \lambda'(B' - \lambda A') + (B - \lambda A) = 0.$$

But since λ' depends upon λ , we must have from the first equation

$$C = 0, \quad C' = 0;$$

that is, the collineation must be affine.

Conversely, since any affine transformation may be written

$$x = Ax' + A'y' + A'', \quad y = Bx' + B'y' + B'',$$

equations (5) become

$$(10) \quad X' = h^2(B'X - A'Y), \quad Y' = h^2(-BX + AY),$$

h being an arbitrary constant; consequently

$$Y = \lambda X$$

is transformed into

$$Y' = \lambda' X' \quad \text{where} \quad \lambda' = -\frac{B - A\lambda}{B' - A'\lambda}.$$

Lastly, consider the affine transformation which transforms all the orbits under one conservative force into those under another conservative force. In this transformation we must have

$$\frac{\partial X'}{\partial y'} = \frac{\partial Y'}{\partial x'}$$

under the condition

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}.$$

Since (10) gives

$$\begin{aligned} \frac{\partial Y'}{\partial x'} &= \frac{\partial Y'}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial Y'}{\partial y} \frac{\partial y}{\partial x'} \\ &= h^2 \left[A \left(-B \frac{\partial X}{\partial x} + A \frac{\partial Y}{\partial x} \right) + B \left(-B \frac{\partial X}{\partial y} + A \frac{\partial Y}{\partial y} \right) \right], \\ \frac{\partial X'}{\partial y'} &= h^2 \left[A' \left(B' \frac{\partial X}{\partial x} - A' \frac{\partial Y}{\partial x} \right) + B' \left(B' \frac{\partial X}{\partial y} - A' \frac{\partial Y}{\partial y} \right) \right], \end{aligned}$$

it should be

$$(AB + A'B') \left(\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} \right) = \{ (A^2 + A'^2) - (B^2 + B'^2) \} \frac{\partial Y}{\partial x}$$

identically; so that

$$AB + A'B' = 0, \quad A^2 + A'^2 = B^2 + B'^2;$$

that is, the affine transformation must belong to the principal group. Conversely the transformation

$$\begin{aligned} x &= \rho(x' \cos \theta + y' \sin \theta + a), \\ y &= \pm \rho(x' \sin \theta - y' \cos \theta + b), \end{aligned} \quad (\rho, a, b \text{ being arbitrary constants})$$

carries over all the orbits under one parallel and conservative force into those under another parallel and conservative force.

Therefore the projective group, the affine group and the principal group may be characterised by invariancy of the orbits under any force⁽¹⁾, those under any parallel force⁽²⁾ and those under any parallel and conservative force⁽³⁾ respectively. Consequently, according to Prof. Klein's principle of classification of geometries, *projective geometry belongs to the general force, affine geometry to the parallel force, and elementary geometry to the parallel and conservative force.*

The *W*-force.

5. I. Consider the motion defined by

$$x_1 = k_1 e^{a_1 t}, \quad x_2 = k_2 e^{a_2 t}, \quad x_3 = k_3 e^{a_3 t},$$

where a 's and k 's are any constants, or

$$(11) \quad x = \frac{k_1}{k_3} e^{at}, \quad y = \frac{k_2}{k_3} e^{\beta t}, \quad (a = a_1 - a_3, \quad \beta = a_2 - a_3).$$

The *orbits* form a system of *W*-curves

$$x_1^{a_2 - a_3} x_2^{a_3 - a_1} x_3^{a_1 - a_2} = \text{const.},$$

or

(1) The characteristic properties of these orbits were given by Prof Kasner, loc. cit..

(2) For the characteristic properties of these orbits, see Ogura, Geometry of the field of central force, Tôhoku Math. Jour., 11 (1917), p. 38-54, especially p. 50.

(3) Any parallel and conservative force is defined by the force function of the form

$$U = \Phi(x + ky), \quad (k, \text{ any const.}).$$

In this case all the cubical duplicatrices in my paper (loc. cit. p. 51) corresponding to all the points of a straight line perpendicular to the force lines are congruent; *this property is characteristic for parallel and conservative force.*

$$(12) \quad x^{a_2-a_3} y^{a_3-a_1} = \text{const.},$$

whose invariant triangle coincides with the fundamental triangle (of homogeneous point coordinates).

Differentiating (11) with respect to time t ,

$$(13) \quad \frac{dx}{dt} = \frac{k_1}{k_3} \alpha e^{\alpha t}, \quad \frac{dy}{dt} = \frac{k_2}{k_3} \beta e^{\beta t},$$

hence

$$\left(\frac{dx}{dt} \right)^{a_2-a_3} \left(\frac{dy}{dt} \right)^{a_3-a_1} = \text{const.},$$

which shows us that the hodographs of the orbits (12) form the W -curves of the same system as before.

Differentiating (13) again

$$\frac{d^2x}{dt^2} = \frac{k_1}{k_3} \alpha^2 e^{\alpha t}, \quad \frac{d^2y}{dt^2} = \frac{k_2}{k_3} \beta^2 e^{\beta t};$$

so that

$$\left(\frac{d^2x}{dt^2} \right)^{a_2-a_3} \left(\frac{d^2y}{dt^2} \right)^{a_3-a_1} = \text{const.}$$

Therefore the hodographs of the orbits form also the W -curves of the same system as before.

Now the coordinates of force have the expressions:

$$(14) \quad p_1 = -\beta^2 y, \quad p_2 = \alpha^2 x, \quad p_3 = (\beta^2 - \alpha^2) xy.$$

Hence the force-envelopes corresponding to the orbits (12) are

$$p_1^{a_3-a_1} p_2^{a_2-a_3} p_3^{a_1-a_2} = \text{const.}^{(1)}$$

in homogeneous line coordinates, which form a system of W -curves different from that of the orbits.

II. The orbits described under this force (14) form the W -curves (12) for the initial conditions:

$$x = \frac{k_1}{k_3}, \quad y = \frac{k_2}{k_3}; \quad \frac{dx}{dt} = \frac{k_1}{k_3} \alpha, \quad \frac{dy}{dt} = \frac{k_2}{k_3} \beta \quad \text{for } t=0.$$

For the sake of brevity this force will be called the W -force with respect to the given fundamental triangle. This is the conservative force whose force-function is

(1) This equation may be written

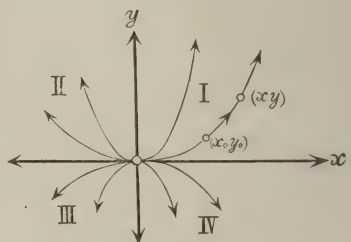
$$x_1^{a_3-a_1} x_2^{a_2-a_3} x_3^{a_1-a_2} = \text{const.}$$

in the point coordinates.

$$\frac{1}{2} (\alpha^2 x^2 + \beta^2 y^2) + \text{const.};$$

so the equipotential curves form a system of concentric and homothetic ellipses. It is noteworthy that the total energy has the same value along all the W -curves (12)(¹).

Here we will prove the fundamental theorem: *Let us divide the plane into the four regions I, II, III, IV as in the figure. Any collineation which leaves the fundamental triangle invariant and transforms one point into another in the same region, can be produced by the W -force.*



Let the collineation be

$$x = \mu x_0, \quad y = \nu y_0,$$

where μ and ν are any positive constants; for (x_0, y_0) and (x, y) lie in the same region. Then if we take

$$\alpha = \frac{\log \mu}{t_1}, \quad \beta = \frac{\log \nu}{t_1},$$

t_1 being considered as any given constant, and $\alpha x_0, \beta y_0$ as the velocity-components at (x_0, y_0) , then the collineation will be produced by the W -force

$$X = \alpha^2 x, \quad Y = \beta^2 y;$$

and the particle at (x_0, y_0) will arrive at (x, y) after the time t_1 .

III. *The field of W -force corresponds to the principal coincidence (1, 1) which is given by*

$$(15) \quad \begin{cases} \beta^2 p_1 x_1 + \alpha^2 p_2 x_2 + (\alpha^2 + \beta^2) p_3 x_3 = 0, \\ p_1 x_1 + p_2 x_2 + p_3 x_3 = 0. \end{cases}$$

And the *lines of force* are the ∞^1 W -curves

$$x_1 - \beta^2 x_2 \alpha^2 x_3 \beta^2 - \alpha^2 = \text{const.},$$

constituting a system different from that of the orbits (12).

Lastly we have from (15)

$$(16) \quad \rho p_1 = -\beta^2 x_2 x_3, \quad \rho p_2 = \alpha^2 x_3 x_1, \quad \rho p_3 = (\beta^2 - \alpha^2) x_1 x_2,$$

ρ being any proportional factor; or

(¹) In the other words, the W -curves (12) belong to a *natural family*.

$$\lambda x_1 = -\beta^2 p_2 p_3, \quad \lambda x_2 = \alpha^2 p_3 p_1, \quad \lambda x_3 = (\beta^2 - \alpha^2) p_1 p_2,$$

λ being any proportional factor. Therefore it follows that *any point and the force-line acting at that point make a one-to-one correspondence which is nothing but an involutory quadratic correlation*⁽¹⁾.

6. Now we will determine the W -force with respect to any fundamental triangle.

I. Let us consider the collineation

$$\begin{cases} \rho x_1 = u_1 x'_1 + u_2 x'_2 + u_3 x'_3 \equiv u_{x'}, \\ \rho x_2 = v_1 x'_1 + v_2 x'_2 + v_3 x'_3 \equiv v_{x'}, \\ \rho x_3 = w_1 x'_1 + w_2 x'_2 + w_3 x'_3 \equiv w_{x'}, \end{cases} \quad (\rho \text{ being any constant}),$$

or

$$\begin{cases} \frac{\Delta}{\rho} x'_1 = U_1 x_1 + V_1 x_2 + W_1 x_3, \\ \frac{\Delta}{\rho} x'_2 = U_2 x_1 + V_2 x_2 + W_2 x_3, \\ \frac{\Delta}{\rho} x'_3 = U_3 x_1 + V_3 x_2 + W_3 x_3, \end{cases}$$

where U_1, V_1, \dots denote the algebraic complements of u_1, v_1, \dots in the determinant

$$\Delta = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

By this collineation the invariant triangle in the last paragraph becomes

$$u_{x'} = 0, \quad v_{x'} = 0, \quad w_{x'} = 0;$$

and the orbits (12) take the form

$$(17) \quad w_{x'}^{a_2 - a_3} v_{x'}^{a_3 - a_1} w_{x'}^{a_1 - a_2} = \text{const.}$$

It is easily seen from (7) that the coordinates of force are transformed into

$$\begin{cases} \sigma p'_1 = u_1 p_1 + v_1 p_2 + w_1 p_3, \\ \sigma p'_2 = u_2 p_1 + v_2 p_2 + w_2 p_3, \\ \sigma p'_3 = u_3 p_1 + v_3 p_2 + w_3 p_3, \end{cases} \quad \sigma = \frac{x_3^2}{k^2 (U_3 x_1 + V_3 x_2 + W_3 x_3)^2};$$

whence we find

(1) See Klein und Lie, Math. Ann., 4 (1871), p. 78.

$$(18) \left\{ \begin{aligned} \frac{1}{k^2 \Delta^2} p'_1 &= \frac{x_3'^2}{w_{x'}^4} \left\{ -\beta^2 u_1 v_{x'} w_{x'} + \alpha^2 v_1 w_{x'} u_{x'} + (\beta^2 - \alpha^2) w_1 u_{x'} v_{x'} \right\}, \\ \frac{1}{k^2 \Delta^2} p'_2 &= \frac{x_3'^2}{w_{x'}^4} \left\{ -\beta^2 u_2 v_{x'} w_{x'} + \alpha^2 v_2 w_{x'} u_{x'} + (\beta^2 - \alpha^2) w_2 u_{x'} v_{x'} \right\}, \\ \frac{1}{k^2 \Delta^2} p'_3 &= \frac{x_3'^2}{w_{x'}^4} \left\{ -\beta^2 u_3 v_{x'} w_{x'} + \alpha^2 v_3 w_{x'} u_{x'} + (\beta^2 - \alpha^2) w_3 u_{x'} v_{x'} \right\} \quad (1). \end{aligned} \right.$$

Similarly the coordinates of velocity take the form

$$\left\{ \begin{aligned} \frac{1}{k} q'_1 &= \frac{1}{w_{x'}^2} \left\{ -\beta u_1 v_{x'} w_{x'} + \alpha v_1 w_{x'} u_{x'} + (\beta - \alpha) w_1 u_{x'} v_{x'} \right\}, \\ \frac{1}{k} q'_2 &= \frac{1}{w_{x'}^2} \left\{ -\beta u_2 v_{x'} w_{x'} + \alpha v_2 w_{x'} u_{x'} + (\beta - \alpha) w_2 u_{x'} v_{x'} \right\}, \\ \frac{1}{k} q'_3 &= \frac{1}{w_{x'}^2} \left\{ -\beta u_3 v_{x'} w_{x'} + \alpha v_3 w_{x'} u_{x'} + (\beta - \alpha) w_3 u_{x'} v_{x'} \right\}. \end{aligned} \right.$$

II. In order to find the motion, take the time transformation

$$k dt' = \frac{dt}{(U_3 x + V_3 y + W_3)^2},$$

which may be written

$$(19) \quad k dt' = \frac{k_3^2 dt}{(k_1 U_3 e^{\alpha t} + k_2 V_3 e^{\beta t} + k_3 W_3)^2};$$

and let the solution of the differential equation (19) be

$$t = \varphi(t') \quad \text{or} \quad t' = \psi(t).$$

Then for the initial conditions

$$\begin{aligned} x' &= \frac{k_1 U_1 + k_2 V_1 + k_3 W_1}{k_1 U_3 + k_2 V_3 + k_3 W_3}, & y' &= \frac{k_1 U_2 + k_2 V_2 + k_3 W_2}{k_1 U_3 + k_2 V_3 + k_3 W_3}, \\ \frac{1}{k} \frac{dx'}{dt'} &= -\frac{k_2}{k_3} \beta u_2 + \frac{k_1}{k_3} \alpha v_2 + \frac{k_1 k_2}{k_3^2} (\beta - \alpha) w_2, \\ \frac{1}{k} \frac{dy'}{dt'} &= \frac{k_2}{k_3} \beta u_1 - \frac{k_1}{k_3} \alpha v_1 - \frac{k_1 k_2}{k_3^2} (\beta - \alpha) w_1, \\ &\text{for } t' = \psi(0), \end{aligned}$$

a particle describes a W -curve (17) under the W -force (18); and the motion is given by

(1) This force is not conservative in general.

$$\begin{cases} x' = \frac{k_1 U_1 e^{at} + k_2 V_1 e^{\beta t} + k_3 W_1}{k_1 U_3 e^{at} + k_2 V_3 e^{\beta t} + k_3 W_3}, \\ y' = \frac{k_1 U_2 e^{at} + k_2 V_2 e^{\beta t} + k_3 W_2}{k_1 U_3 e^{at} + k_2 V_3 e^{\beta t} + k_3 W_3}, \end{cases}$$

i.e.

$$\begin{cases} \lambda x_1' = k_1 U_1 e^{a_1 \varphi(t')} + k_2 V_1 e^{a_2 \varphi(t')} + k_3 W_1 e^{a_3 \varphi(t')}, \\ \lambda x_2' = k_1 U_2 e^{a_1 \varphi(t')} + k_2 V_2 e^{a_2 \varphi(t')} + k_3 W_2 e^{a_3 \varphi(t')}, \\ \lambda x_3' = k_1 U_3 e^{a_1 \varphi(t')} + k_2 V_3 e^{a_2 \varphi(t')} + k_3 W_3 e^{a_3 \varphi(t')}. \end{cases}$$

The fundamental theorem in § 5, II. is also true in the general case if the plane is divided into the four regions as in the figure.

III. The force-envelopes corresponding to the orbits (17) are

$$U_{p'}^{a_3} - a_1 V_{p'}^{a_2} - a_3 W_{p'}^{a_1} = \text{const.},$$

which form a system of W -curves different from that of the orbits.

The principal coincidence corresponding to the W -force (18) is determined by

$$\begin{cases} \beta^2 p_1' u_{x'} + a^2 p_2' v_{x'} + (a^2 + \beta^2) p_3' w_{x'} = 0, \\ p_1' u_{x'} + p_2' v_{x'} + p_3' w_{x'} = 0; \end{cases}$$

so that the lines of force are

$$u_{x'}^{-\beta^2} v_{x'}^{a^2} w_{x'}^{\beta^2 - a^2} = \text{const.},$$

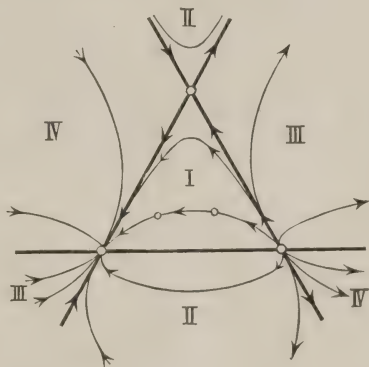
which form a system of W -curves different from that of the orbits.

In the particular case where

$$\begin{aligned} u_1 &= 1, & u_2 &= \sqrt{-1}, & u_3 &= 0, \\ v_1 &= 1, & v_2 &= -\sqrt{-1}, & v_3 &= 0, \\ w_1 &= 0, & w_2 &= 0, & w_3 &= 1, \\ \alpha &= \frac{1}{2}(1 + \sqrt{-1} \cdot \cot \omega), & \beta &= \frac{1}{2}(1 - \sqrt{-1} \cdot \cot \omega), & \gamma &= -1, \end{aligned}$$

the orbits (17) become a family of (real) logarithmic spirals

$$r = C \cdot e^{\theta \cot \omega}$$



in the polar coordinates (r, θ) , where

$$x' = r \cos \theta, \quad y' = r \sin \theta.$$

Then the W -force becomes imaginary⁽¹⁾:

$$p_1' = -\sqrt{-1} k^2 \{2 \cot \omega \cdot x' + (1 - \cot^2 \omega) y'\},$$

$$p_2' = -\sqrt{-1} k^2 \{(\cot^2 \omega - 1)x' + 2 \cot \omega \cdot y'\},$$

$$p_3' = 2\sqrt{-1} k^2 \cot \omega (x'^2 + y'^2).$$

On the contrary the lines of force form a family of real logarithmic spirals

$$r = c' \cdot e^{\theta \cot(-2\omega)}.$$

7. [Appendix]. Here I take this opportunity to determine the central force under which a system of W -curves (of the first kind) can be described.

Let the equation of a system of W -curves be

$$\xi_1^{\lambda_1} \xi_2^{\lambda_2} \xi_3^{\lambda_3} = \text{const.} \quad (\lambda_1 + \lambda_2 + \lambda_3 = 0),$$

or

$$f \equiv \lambda_1 \log \xi_1 + \lambda_2 \log \xi_2 + \lambda_3 \log \xi_3 = \text{const.},$$

where we have put

$$\xi_1 \equiv u_1 x + u_2 y + u_3, \quad \xi_2 \equiv v_1 x + v_2 y + v_3, \quad \xi_3 \equiv w_1 x + w_2 y + w_3.$$

Take the centre of force as the origin $(x=0, y=0)$; and let h be the angular momentum about the origin, and the equations of motion be

$$\frac{d^2 x}{dt^2} = F \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{d^2 y}{dt^2} = F \frac{y}{\sqrt{x^2 + y^2}}.$$

Then the required force F is given by

$$F = -h^2 \sqrt{x^2 + y^2} \cdot \frac{\left(\frac{\partial f}{\partial y}\right)^2 \frac{\partial^2 f}{\partial x^2} - 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial y} + \left(\frac{\partial f}{\partial x}\right)^2 \frac{\partial^2 f}{\partial y^2}}{\left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}\right)^3} \quad (^2).$$

Now by a short calculation

(¹) In this case the time and the velocity become imaginary. Compare with Appell, Sur une interprétation des valeurs imaginaire du temps en mécanique, Comptes rendus, Paris, 87 (1878), p. 1074.

(²) Whittaker, Treatise on the analytical dynamics (1904), p. 78.

$$\begin{aligned}
 & \left(\frac{\partial f}{\partial y} \right)^2 \frac{\partial^2 f}{\partial x^2} - 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x \partial y} + \left(\frac{\partial f}{\partial x} \right)^2 \frac{\partial^2 f}{\partial y^2} \\
 &= \frac{\lambda_1 \lambda_2 \lambda_3}{\xi_1^2 \xi_2^2 \xi_3^2} (U_3 \xi_1 + V_3 \xi_2 + W_3 \xi_3)^2 \\
 &= \frac{\lambda_1 \lambda_2 \lambda_3}{(\xi_1 \xi_2 \xi_3)^2} \Delta^2, \\
 & x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = - \frac{1}{\xi_1 \xi_2 \xi_3} (\lambda_1 u_3 \xi_2 \xi_3 + \lambda_2 v_3 \xi_3 \xi_1 + \lambda_3 w_3 \xi_1 \xi_2);
 \end{aligned}$$

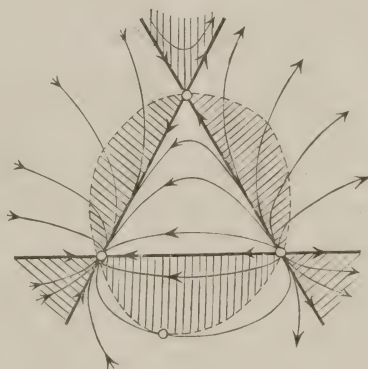
and therefore

$$F = h^2 J^2 \lambda_1 \lambda_2 \lambda_3 \sqrt{x^2 + y^2} \frac{\xi_1 \xi_2 \xi_3}{(\lambda_1 u_3 \xi_2 \xi_3 + \lambda_2 v_3 \xi_3 \xi_1 + \lambda_3 w_3 \xi_1 \xi_2)^3}.$$

The force vanishes on the invariant lines and becomes infinite on the conic

$$\lambda_1 u_3 \xi_2 \xi_3 + \lambda_2 v_3 \xi_3 \xi_1 + \lambda_3 w_3 \xi_1 \xi_2 = 0,$$

which is the locus of points of contact of all the tangents drawn from the origin to the system of W -curves. And it is easily seen that this conic passes through three vertices of the invariant triangle and the origin, and touches the W -curve belonging to the given system at the origin. In the annexed figure the central force is attractive or repulsive according as the particle is outside or inside the shaded regions.



Particularly,

(i) if we put

$$\xi_3 = 1, \quad \lambda_1 = 1, \quad \lambda_2 = 1,$$

we have

$$\xi_1 \xi_2 = c,$$

so that the system consists of the concentric and homothetic conics whose asymptotes are $\xi_1 = 0$, $\xi_2 = 0$. Then

$$(20) \quad F = k \sqrt{x^2 + y^2} (v_3 \xi_1 + u_3 \xi_2 - 2c)^{-3}, \quad (k, \text{ a const.})$$

which is nothing but Hamilton's theorem⁽¹⁾.

(ii) Next if we put

$$\lambda_1=1, \quad \lambda_2=1,$$

we have

$$\xi_1 \xi_2 = \text{const.} \quad \xi_3^2,$$

so that the system consists of the conics having the double contact whose tangents at the points of contact are $\xi_1=0$ and $\xi_2=0$. Then

$$F=k \sqrt{x^2+y^2} (\xi_1 \xi_2)^{-\frac{3}{2}}.$$

More particularly, when the vertex ($\xi_1=0, \xi_2=0$) coincides with the origin, the force takes the form

$$(21) \quad F=k \sqrt{x^2+y^2} (ax^2+2\beta xy+\gamma y^2)^{-\frac{3}{2}}.$$

Thus we have arrived at the two laws (20), (21) of central force discovered by Darboux and Halphen in Bertrand's problem.

(iii) Lastly if we put

$$\begin{aligned} \xi_1 &= (x-a) + \sqrt{-1}(y-\beta), & \xi_2 &= (x-a) - \sqrt{-1}(y-\beta), & \xi_3 &= 1, \\ \lambda_1 &= -\frac{1}{2}(1 + \sqrt{-1} \cot \omega), & \lambda_2 &= \frac{1}{2}(1 - \sqrt{-1} \cot \omega), & \lambda_3 &= -1, \end{aligned}$$

we have a system of logarithmic spirals with the same pole (a, β) and the same angle ω (ω being the angle between the radius vector and the curve). In this case we have

$$\begin{aligned} F &= -\frac{h^2}{\sin^2 \omega} \sqrt{x^2+y^2} \\ &\times \frac{(x-a)^2 + (y-\beta)^2}{\{(x-a)^2 + (y-\beta)^2 + (a-\beta \cot \omega)(x-a) + (\beta + a \cot \omega)(y-\beta)\}^3}, \end{aligned}$$

so that the force is attractive or repulsive according as a particle is outside or inside the circle

$$(x-a)^2 + (y-\beta)^2 + (a-\beta \cot \omega)(x-a) + (\beta + a \cot \omega)(y-\beta) = 0,$$

which is the locus of points of contact of all the tangents drawn from the centre of force to the system of logarithmic spirals⁽²⁾.

(1) Hamilton, Proc. Roy. Irish Acad. (1846).

(2) This circle passes through the centre of force and the common pole of logarithmic spirals, and touches the logarithmic spiral through the centre at that point.

When the centre of force coincides with the pole, the force takes the form

$$F = -\frac{h^2}{\sin^2 \omega} (x^2 + y^2)^{-\frac{3}{2}}$$

which is a well known result.

PART II. MOTION IN SPACE.

General force.

8. Consider the field of force defined by

$$\frac{d^2x}{dt^2} = X(x, y, z), \quad \frac{d^2y}{dt^2} = Y(x, y, z), \quad \frac{d^2z}{dt^2} = Z(x, y, z);$$

and let us introduce the *coordinates of force*:

$$(22) \quad \begin{cases} p_{12} = xY - yX, & p_{23} = yZ - zY, & p_{31} = zX - xZ, \\ p_{41} = X, & p_{42} = Y, & p_{43} = Z, \\ p_{ij} = -p_{ji}, & (i, j = 1, 2, 3, 4), \end{cases}$$

where p_{41}, p_{42}, p_{43} are the components of the force parallel to the coordinate axes and p_{12}, p_{23}, p_{31} the moments of the force about the axes. If we use the homogeneous point coordinates such that

$$x = \frac{x_1}{x_4}, \quad y = \frac{x_2}{x_4}, \quad z = \frac{x_3}{x_4},$$

we have the identities (*relation of incidence*)

$$\begin{cases} p_{34}x_2 + p_{42}x_3 + p_{23}x_4 = 0, \\ p_{43}x_1 + p_{14}x_3 + p_{31}x_4 = 0, \\ p_{24}x_1 + p_{41}x_2 + p_{12}x_4 = 0, \\ p_{32}x_1 + p_{13}x_2 + p_{21}x_3 = 0, \end{cases}$$

and

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

Hence (p_{ij}) may be considered as homogeneous line coordinates (in the sense of Plücker) of the force-line acting at (x, y, z) .

When the point (x_i) describes an orbit, the force-line (p_{ij}) envelopes

a ruled surface which may be called the *force-envelope* corresponding to the orbit.

If a point be given in the field, there is one force-line acting at that point; so that the totality of force-lines must form, in general⁽¹⁾, a line complex

$$(23) \quad \Omega(p_{12}, p_{23}, p_{31}, p_{41}, p_{42}, p_{43}) = 0$$

which is equivalent to equations (22), x, y, z being considered as three parameters. Conversely, if a line belonging to the complex (23) be given, there are, in general, some points where the acting lines coincide with the given line. Hence *any field gives a certain correspondence between the points in space and the lines of a complex.*

9. The theory of point-line connexes⁽²⁾ leads us to the construction of fields of force.

Let $\pi_{12}, \pi_{13}, \pi_{14}, \pi_{23}, \pi_{42}, \pi_{34}$ be the homogeneous line coordinates of Plücker. Then we have the identity

$$\pi_{12} \pi_{34} + \pi_{13} \pi_{42} + \pi_{14} \pi_{23} = 0.$$

Now consider the equation of the form (which will be called the *point-line connex* of the first order and n^{th} class)

$$(24) \quad \pi_{12} f_{12}^m + \pi_{23} f_{23}^m + \pi_{31} f_{31}^m + \pi_{41} f_{41}^m + \pi_{42} f_{42}^m + \pi_{43} f_{43}^m = 0,$$

where all f^m 's are homogeneous polynomials of the n^{th} degree with respect to x_1, x_2, x_3, x_4 ; and the four equations of incidence

$$(25) \quad \begin{cases} \pi_{34} x_2 + \pi_{42} x_3 + \pi_{23} x_4 = 0, \\ \pi_{43} x_1 + \pi_{14} x_3 + \pi_{31} x_4 = 0, \\ \pi_{24} x_1 + \pi_{41} x_2 + \pi_{12} x_4 = 0, \\ \pi_{32} x_1 + \pi_{13} x_2 + \pi_{21} x_3 = 0. \end{cases}$$

Taking (x_i) to be any given point, equations (24) and (25) give the corresponding lines which constitute a flat pencil having the given point as the vertex; conversely, taking (p_{ij}) to be any given line, the corresponding m points are finite in number, except a certain cases.

Next consider the two equations

(1) There exists the case where the totality of all the force-lines form a *congruence*, as for the central force. In what follows we will not consider this exceptional case unless the contrary is stated.

(2) For a detailed study of this theory, see my paper referred to in the introduction.

$$(26) \quad \begin{cases} \pi_{12}f_{12}^m + \pi_{22}f_{23}^m + \pi_{31}f_{31}^m + \pi_{41}f_{41}^m + \pi_{42}f_{42}^m + \pi_{43}f_{43}^m = 0, \\ \pi_{12}f_{12}^n + \pi_{23}f_{23}^n + \pi_{31}f_{31}^n + \pi_{41}f_{41}^n + \pi_{42}f_{42}^n + \pi_{43}f_{43}^n = 0 \end{cases}$$

and (25). Taking (x_i) to be any given point, it corresponds to only one line passing through that point; and hence the totality of the lines (π_{ij}) must form a complex

$$(27) \quad \Omega(\pi_{12}, \pi_{23}, \pi_{31}, \pi_{41}, \pi_{42}, \pi_{43}) = 0.$$

Now it will be seen, by the aid of the identity

$$\pi_{12}\pi_{34} + \pi_{13}\pi_{42} + \pi_{14}\pi_{23} = 0,$$

that the determinant

$$\begin{vmatrix} 0 & \pi_{34} & \pi_{42} & \pi_{23} \\ \pi_{43} & 0 & \pi_{14} & \pi_{31} \\ \pi_{24} & \pi_{41} & 0 & \pi_{12} \\ \pi_{32} & \pi_{13} & \pi_{21} & 0 \end{vmatrix}$$

is of rank 2. Hence the necessary and sufficient condition that the six equations (25) and (26) should be consistent for the given (π_{ij}) will be obtained by eliminating x_1, x_2, x_3, x_4 from the four homogeneous equations: (26) and any two of (25). This equation of the condition is nothing but the equation of the complex (27)⁽¹⁾.

Consequently, taking (π_{ij}) to be any given line belonging to the complex (27), it corresponds to some points lying on that line. Thus we have established a correspondence between the points of space and the lines of the complex (27).

Solving six equations (25) and (26) we find

$$(28) \quad \begin{cases} \rho\pi_{41} = \begin{vmatrix} f_{12}^m x_1 - f_{23}^m x_3 + f_{42}^m x_4 & -f_{31}^m x_1 + f_{23}^m x_2 + f_{43}^m x_4 \\ f_{12}^n x_1 - f_{23}^n x_3 + f_{42}^n x_4 & -f_{31}^n x_1 + f_{23}^n x_2 + f_{43}^n x_4 \end{vmatrix} \equiv x_2^{m+n+2} \varphi_{41}, \\ \rho\pi_{42} = \begin{vmatrix} -f_{31}^m x_1 + f_{23}^m x_2 + f_{43}^m x_4 & -f_{12}^m x_2 + f_{31}^m x_3 + f_{41}^m x_4 \\ -f_{31}^n x_1 + f_{23}^n x_2 + f_{43}^n x_4 & -f_{12}^n x_2 + f_{31}^n x_3 + f_{41}^n x_4 \end{vmatrix} \equiv x_4^{m+n+2} \varphi_{42}, \\ \rho\pi_{43} = \begin{vmatrix} -f_{12}^m x_2 + f_{31}^m x_3 + f_{41}^m x_4 & f_{12}^m x_1 - f_{23}^m x_3 + f_{42}^m x_4 \\ -f_{12}^n x_2 + f_{31}^n x_3 + f_{41}^n x_4 & f_{12}^n x_1 - f_{23}^n x_3 + f_{42}^n x_4 \end{vmatrix} \equiv x_4^{m+n+2} \varphi_{43}, \\ \rho\pi_{12} = \dots \equiv x_4^{m+n+2} \varphi_{12}, \quad \rho\pi_{23} = \dots \equiv x_4^{m+n+2} \varphi_{23}, \quad \rho\pi_{31} = \dots \equiv x_4^{m+n+2} \varphi_{31}, \end{cases}$$

(1) In general, this complex is shown to be of the $(m+n+1)$ th degree.

all φ 's being functions of x, y, z ; whence we obtain the infinitely many fields of force defined by

$$(29) \quad \begin{cases} X = \lambda \varphi_{41}, & Y = \lambda \varphi_{42}, & Z = \lambda \varphi_{43}, \\ xY - yX = \lambda \varphi_{12}, & yZ - zY = \lambda \varphi_{23}, & zX - xZ = \lambda \varphi_{31}. \end{cases}$$

(λ being taken as an arbitrary function of x, y, z) in which the totality of the force-lines form the complex (27).

10. When the field is given by (22), the ∞^2 lines of force satisfy the two differential equations

$$(30) \quad \frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} \quad (1),$$

so that the ∞^2 lines of force are the characteristics of the linear partial differential equation of the first order

$$(31) \quad F \equiv X \frac{\partial z}{\partial x} + Y \frac{\partial z}{\partial y} - Z = 0;$$

and since the lines of force are integral curves of the line complex

$$(23) \quad \Omega(p_{12}, p_{23}, p_{31}, p_{41}, p_{42}, p_{43}) = 0,$$

the tangent to any characteristic is a generator of the complex cone corresponding to the point of contact.

Now let us take a surface which touches corresponding complex cone at any point of it. All these surfaces are given by the partial

(1) If the field be defined by (28), we have

$$\frac{xdy - ydx}{\pi_{12}} = \frac{ydz - zdy}{\pi_{23}} = \frac{zdx - xdz}{\pi_{31}} = \frac{dx}{\pi_{41}} = \frac{dy}{\pi_{42}} = \frac{dz}{\pi_{43}}$$

along the lines of force. Hence (26) become

$$f_{12}^m (xdy - ydx) + \dots + f_{43}^m dz = 0, \quad f_{12}^n (xdy - ydx) + \dots + f_{43}^n dz = 0,$$

from which we obtain the differential equations of the lines of force

$$\begin{aligned} \frac{dx}{\left| \begin{array}{cc} f_{12}^m x - f_{23}^m z + f_{42}^m & f_{23}^m y - f_{31}^m x + f_{43}^m \\ f_{12}^n x - f_{23}^n z + f_{42}^n & f_{23}^n y - f_{31}^n x + f_{43}^n \end{array} \right|} &= \frac{dy}{\left| \begin{array}{cc} f_{23}^m y - f_{31}^m x + f_{43}^m & f_{31}^m z - f_{12}^m y + f_{41}^m \\ f_{23}^n y - f_{31}^n x + f_{43}^n & f_{31}^n z - f_{12}^n y + f_{41}^n \end{array} \right|} \\ &= \frac{dz}{\left| \begin{array}{cc} f_{31}^m z - f_{12}^m y + f_{41}^m & f_{12}^m x - f_{23}^m z + f_{42}^m \\ f_{31}^n z - f_{12}^n y + f_{41}^n & f_{12}^n x - f_{23}^n z + f_{42}^n \end{array} \right|}. \end{aligned}$$

It is easily seen from (28), (29) that this system is identical with (30).

differential equation⁽¹⁾

$$(32) \quad G\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$$

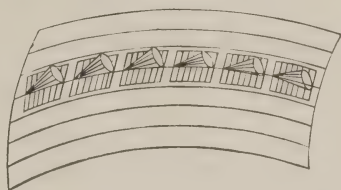
which is obtained by eliminating x', y', z' and ρ from

$$\Omega(yz' - zy', zx' - xz', xy' - yx', x', y', z') = 0,$$

$$\rho \frac{\partial z}{\partial x} = \frac{\partial \Omega}{\partial x'}, \quad \rho \frac{\partial z}{\partial y} = \frac{\partial \Omega}{\partial y'}, \quad -\rho = \frac{\partial \Omega}{\partial z'}.$$

The tangent to any characteristic of the partial differential equation (32) coincides with the generator of the complex cone along which the cone touches the integral (complete) surface.

Consequently in order that there may exist an integral surface S of the differential equation (32) on which the lines of force should be the characteristics



of that differential equation, that is, in order that the complex cones may touch a surface S along the ∞^1 lines of force on that surface, it is necessary and sufficient that the two partial differential equations (31) and (32) should have the common integral; in other words, $F=0$ and $G=0$ should be in involution:

$$\left| \begin{array}{cc} \frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z} & \frac{\partial G}{\partial x} + p \frac{\partial G}{\partial z} \\ \frac{\partial F}{\partial p} & \frac{\partial G}{\partial p} \end{array} \right| + \left| \begin{array}{cc} \frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z} & \frac{\partial G}{\partial y} + q \frac{\partial G}{\partial z} \\ \frac{\partial F}{\partial q} & \frac{\partial G}{\partial q} \end{array} \right| = 0,$$

$$\left(p \equiv \frac{\partial z}{\partial x}, \quad q \equiv \frac{\partial z}{\partial y} \right),$$

under $F=0$, $G=0$. This condition may be written, in virtue of

$$(31) \quad F \equiv Xp + Yq - Z = 0,$$

in the form

$$X \frac{\partial G}{\partial x} + Y \frac{\partial G}{\partial y} + Z \frac{\partial G}{\partial z} + P \frac{\partial G}{\partial p} + Q \frac{\partial G}{\partial q} = 0 \quad \text{whenever } G=0,$$

where we have put

(1) Lie, Geometrie der Berührungstransformationen, I (1896), p. 260.

$$P = \frac{\partial Z}{\partial x} + p \frac{\partial Z}{\partial z} - p \left(\frac{\partial X}{\partial x} + p \frac{\partial X}{\partial z} \right) - q \left(\frac{\partial Y}{\partial x} + p \frac{\partial Y}{\partial z} \right),$$

$$Q = \frac{\partial Z}{\partial y} + q \frac{\partial Z}{\partial z} - p \left(\frac{\partial X}{\partial y} + q \frac{\partial X}{\partial z} \right) - q \left(\frac{\partial Y}{\partial y} + q \frac{\partial Y}{\partial z} \right);$$

which is nothing but the necessary and sufficient condition that the differential equation

$$(32) \quad G(x, y, z, p, q) = 0$$

may be invariant for the infinitesimal transformation

$$Uf \equiv X \frac{\partial f}{\partial x} + Y \frac{\partial f}{\partial y} + Z \frac{\partial f}{\partial z} \quad (1).$$

But since the lines of force

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$$

are the path-curves of that infinitesimal transformation, we have the theorem:

The necessary and sufficient condition that the surface S may exist in the field of force (22) is that the differential equation (32) should be invariant for the infinitesimal transformation

$$Uf \equiv X \frac{\partial f}{\partial x} + Y \frac{\partial f}{\partial y} + Z \frac{\partial f}{\partial z}$$

whose path-curves are the lines of force.

If the condition be satisfied, the common complete integral contains one arbitrary constant, so that there exist ∞^1 surfaces $S^{(2)}$. Then we can infer, by a theorem due to Lie⁽³⁾, the following theorem. *All the lines of force on the surface S (and on this surface only) constitute a system of the asymptotic lines.*

The existence of the surface S depends upon the nature of the field. In § 15 we will show that ∞^1 surfaces S may exist for the W -force.

(1) Lie, loc. cit., 598-600; p. 610.

(2) If the complex $\Omega=0$ (or more generally all the curves of complex) be invariant for the above infinitesimal transformation, it follows from the method of formation of (32) that this differential equation $G=0$ is also invariant. Hence there exist ∞^1 surfaces S , if the infinitesimal transformation

$$Uf \equiv X \frac{\partial f}{\partial x} + Y \frac{\partial f}{\partial y} + Z \frac{\partial f}{\partial z}$$

leave invariant the complex $\Omega=0$ (or more generally all the curves of the complex.)

(3) Lie, loc cit., p. 308.

Here we will give an example of the field, for which the surface S does not exist.

Consider the force defined by

$X=x(b-c+\sqrt{z})$, $Y=y(a-c+\sqrt{z})$, $Z=\sqrt{z}(a-c+\sqrt{z})(b-c+\sqrt{z})$, where \sqrt{z} denotes the positive square root of z and a, b, c are any constants. Then the totality of the force-lines form the tetrahedral complex

$$ap_{23}p_{41}+bp_{31}p_{42}+cp_{12}p_{43}=0.$$

Since the lines of force are given by

$$\frac{dx}{x(b-c+\sqrt{z})}=\frac{dy}{y(a-c+\sqrt{z})}=\frac{dz}{\sqrt{z}(a-c+\sqrt{z})(b-c+\sqrt{z})},$$

they are the ∞^2 conics

$$x=c_1(a+\tau)^2, \quad y=c_2(b+\tau)^2, \quad z=(c+\tau)^2,$$

where τ denotes the parameter and c_1, c_2 are arbitrary constants. In order that the surface S may exist in this case, ∞^1 conics must form a system of the asymptotic lines, which is not true; for every plane asymptotic line is a straight line.

More generally, the surface S can not exist for any field in which all the lines of force are plane curves⁽¹⁾.

11. We can show that the *Appell transformation in space*

$$\left\{ \begin{array}{l} \lambda x_1' = a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + a_{41}x_4, \\ \lambda x_2' = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4, \\ \lambda x_3' = a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4, \\ \lambda x_4' = a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4, \\ k dt' = \frac{x_4^2 dt}{(a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4)^2} \end{array} \right.$$

carries over the given field into the new field defined by

$$\left\{ \begin{array}{l} \rho p'_{41} = \left| \begin{array}{cc} a_{11} & a_{41} \\ a_{14} & a_{44} \end{array} \right| p_{41} + \left| \begin{array}{cc} a_{21} & a_{41} \\ a_{24} & a_{44} \end{array} \right| p_{42} + \left| \begin{array}{cc} a_{31} & a_{41} \\ a_{34} & a_{44} \end{array} \right| p_{43} \\ \quad + \left| \begin{array}{cc} a_{11} & a_{21} \\ a_{14} & a_{24} \end{array} \right| p_{21} + \left| \begin{array}{cc} a_{11} & a_{31} \\ a_{14} & a_{34} \end{array} \right| p_{31} + \left| \begin{array}{cc} a_{21} & a_{31} \\ a_{24} & a_{34} \end{array} \right| p_{32}, \\ \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \end{array} \right.$$

(1) We exclude any field in which all the lines of force are straight lines; for in such a field the totality of all the force-lines form a congruence.

where

$$\rho = \frac{x_4^2}{k^2(a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4)^2}.$$

The transformation coordinates of velocity-line in space and the classification of space geometries are furnished by similar ways as in the plane.

The W -force.

12. Now consider the motion defined by

$$x_1 = k_1 e^{a_1 t}, \quad x_2 = k_2 e^{a_2 t}, \quad x_3 = k_3 e^{a_3 t}, \quad x_4 = k_4 e^{a_4 t},$$

all a 's and k 's being any constants; or

$$x = A e^{a t}, \quad y = B e^{\beta t}, \quad z = C e^{\gamma t},$$

where

$$A = \frac{k_1}{k_4}, \quad B = \frac{k_2}{k_4}, \quad C = \frac{k_3}{k_4},$$

$$a = a_1 - a_4, \quad \beta = a_2 - a_4, \quad \gamma = a_3 - a_4.$$

The orbits form a system of ∞^2 W -curves

$$(33) \quad x^{\frac{1}{a}} : y^{\frac{1}{\beta}} : z^{\frac{1}{\gamma}} = A_1 : B_1 : C_1,$$

(A_1, B_1, C_1 being arbitrary constants), whose invariant tetrahedron coincides with the fundamental tetrahedron of homogeneous coordinates.

Since

$$\frac{dx}{dt} = A a e^{a t} = a x, \quad \dots \quad \dots;$$

$$\frac{d^2 x}{dt^2} = A a^2 e^{a t} = a^2 x, \quad \dots \quad \dots,$$

the *hodographs* of the orbits are the W -curves belonging to the same system as the orbits, and also the *hodographs of hodographs* of the orbits are of the same nature.

The force (the W -force with respect to the given fundamental tetrahedron)

$$(34) \quad X = a^2 x, \quad Y = \beta^2 y, \quad Z = \gamma^2 z$$

is the conservative force having the force function

$$\frac{1}{2} (\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2) + \text{const.};$$

so that the equipotential surfaces form a system of concentric and homothetic ellipsoids; and the total energy has the same value along all the orbits (33).

The *lines of force* are determined by the two differential equations

$$\frac{dx}{\alpha^2 x} = \frac{dy}{\beta^2 y} = \frac{dz}{\gamma^2 z},$$

which give by integration

$$\frac{1}{x\alpha^2} : \frac{1}{y\beta^2} : \frac{1}{z\gamma^2} = A_2 : B_2 : C_2,$$

so that they constitute a system of ∞^2 *W*-curves different from that of the orbits (33).

It is evident that an analogue to the fundamental theorem in § 5, II. is also true, if the space be divided into the eight regions by the three coordinate planes.

Lastly we remark that the *W*-force with respect to the general tetrahedron can be determined in a similar manner as in § 6.

13. Now we will discuss the *force-envelope* of the *W*-force (34).

Let x, y, z be any point of the force-envelope. Then we have

$$(35) \quad \begin{cases} x = A e^{\alpha t} + \frac{A \alpha^2 e^{\alpha t}}{\sqrt{A^2 \alpha^4 e^{2\alpha t} + B^2 \beta^4 e^{2\beta t} + C^2 \gamma^4 e^{2\gamma t}}} \tau, \\ y = B e^{\beta t} + \frac{B \beta^2 e^{\beta t}}{\sqrt{A^2 \alpha^4 e^{2\alpha t} + B^2 \beta^4 e^{2\beta t} + C^2 \gamma^4 e^{2\gamma t}}} \tau, \\ z = C e^{\gamma t} + \frac{C \gamma^2 e^{\gamma t}}{\sqrt{A^2 \alpha^4 e^{2\alpha t} + B^2 \beta^4 e^{2\beta t} + C^2 \gamma^4 e^{2\gamma t}}} \tau, \end{cases}$$

t, τ being the parameters. If we put

$$u = t, \quad v = \frac{\tau}{\sqrt{A^2 \alpha^4 e^{2\alpha t} + B^2 \beta^4 e^{2\beta t} + C^2 \gamma^4 e^{2\gamma t}}},$$

(35) becomes

$$(36) \quad \begin{cases} x = A e^{\alpha u} (1 + \alpha^2 v), \\ y = B e^{\beta u} (1 + \beta^2 v), \\ z = C e^{\gamma u} (1 + \gamma^2 v). \end{cases}$$

The parametric curves $u = \text{const.}$ form the system of generating lines.

But since

$$\begin{aligned} p_{41} &= ax, & p_{42} &= by, & p_{43} &= cz, \\ p_{12} &= (b-a)xy, & p_{23} &= (c-b)yz, & p_{31} &= (a-c)zx, \end{aligned}$$

where

$$a = \alpha^2 = (a_1 - a_4)^2, \quad b = \beta^2 = (a_2 - a_4)^2, \quad c = \gamma^2 = (a_3 - a_4)^2,$$

we obtain

$$(37) \quad bc p_{23} p_{41} + ca p_{32} p_{42} + ab p_{12} p_{43} = 0.$$

Therefore the generating lines belong to the tetrahedral complex (37) with respect to the same fundamental tetrahedron as that of the orbits (33).

Since (36) may be written

$$x^{\frac{1}{\alpha}} : y^{\frac{1}{\beta}} : z^{\frac{1}{\gamma}} = \{A(1 + \alpha^2 v)\}^{\frac{1}{\alpha}} : \{B(1 + \beta^2 v)\}^{\frac{1}{\beta}} : \{C(1 + \gamma^2 v)\}^{\frac{1}{\gamma}},$$

the other parametric curves $v = \text{const.}$ form a system of W -curves, belonging to any tetrahedral complex

$$\lambda p_{23} p_{41} + \mu p_{31} p_{42} + \nu p_{12} p_{43} = 0,$$

where λ, μ, ν satisfy only the condition

$$\frac{\mu - \nu}{\alpha} + \frac{\nu - \lambda}{\beta} + \frac{\lambda - \mu}{\gamma} = 0.$$

The force-envelope is generated by a system of ∞^1 W -curves and may be written in the form

$$z = x^{\frac{\gamma}{\alpha}} \Psi \left(\frac{y^{\alpha}}{x^{\beta}} \right);$$

so it is invariant for the infinitesimal projective transformation

$$Uf \equiv \alpha x \frac{\partial f}{\partial x} + \beta y \frac{\partial f}{\partial y} + \gamma z \frac{\partial f}{\partial z}.$$

Therefore the asymptotic lines of this surface can be found by quadratures⁽¹⁾: in fact, the differential equation of the (curvilinear) asymptotic lines is

$$2 \frac{dv}{du} = \alpha \beta \gamma v^2 - (\alpha + \beta + \gamma) v;$$

which gives by integration

(¹) Lie, Vorlesungen über Differentialgleichungen, (1891), p. 257.

$$\text{const. } e^u = v^{-\frac{2}{a+\beta+\gamma}} \left(v - \frac{a+\beta+\gamma}{a\beta\gamma} \right)^{\frac{2}{a+\beta+\gamma}},$$

or

$$v = \frac{a+\beta+\gamma}{a\beta\gamma} \cdot \left(1 - \text{const. } e^{\frac{a+\beta+\gamma}{2}u} \right)^{-1}.$$

Lastly we add the equation to the line of striction :

$$\begin{aligned} A^2 B^2 \alpha^3 \beta^3 (a-\beta)^2 e^{2(a+\beta)u} (1 + a\beta v) + B^2 C^2 \beta^3 \gamma^3 (\beta-\gamma)^2 e^{2(\beta+\gamma)u} (1 + \beta\gamma v) \\ + C^2 A^2 \gamma^3 \alpha^3 (\gamma-\alpha)^2 e^{2(\gamma+\alpha)u} (1 + \gamma\alpha v) = 0. \end{aligned}$$

14. In order to discuss the configuration of points and force-lines of the field of the W -force (34), consider the six equations

$$(38) \quad \left\{ \begin{array}{l} (bx_2 - cx_3)\pi_{41} + (cx_3 - ax_1)\pi_{42} + (ax_1 - bx_2)\pi_{43} = 0, \\ a(b^2x_2 - c^2x_3)\pi_{41} + b(c^2x_3 - a^2x_1)\pi_{42} + c(a^2x_1 - b^2x_2)\pi_{43} = 0, \\ \pi_{34}x_2 + \pi_{42}x_3 + \pi_{23}x_4 = 0, \\ \pi_{43}x_1 + \pi_{14}x_3 + \pi_{31}x_4 = 0, \\ \pi_{24}x_1 + \pi_{41}x_2 + \pi_{12}x_4 = 0, \\ \pi_{32}x_1 + \pi_{13}x_2 + \pi_{21}x_3 = 0. \end{array} \right.$$

Let (x_i) be a given point. Then solving these equations we find

$$(39) \quad \left\{ \begin{array}{l} \rho\pi_{41} = ax_1x_4, \quad \rho\pi_{42} = bx_2x_4, \quad \rho\pi_{43} = cx_3x_4, \\ \rho\pi_{12} = (b-a)x_1x_2, \quad \rho\pi_{23} = (c-b)x_2x_3, \quad \rho\pi_{31} = (a-c)x_3x_1, \end{array} \right.$$

where ρ is an arbitrary proportional factor. Consequently (π_{ij}) must belong to the tetrahedral complex

$$(40) \quad bc\pi_{23}\pi_{41} + ca\pi_{31}\pi_{42} + ab\pi_{12}\pi_{43} = 0.$$

Conversely, let (π_{ij}) be a given line. In order that the six equations (38) may be consistent, the line (π_{ij}) must belong to the tetrahedral complex (40) ⁽¹⁾. If the condition be fulfilled, we may take

$$(41) \quad \begin{array}{ll} \lambda x_1 = bc\pi_{23}\pi_{41}, & \lambda x_2 = ca\pi_{31}\pi_{42}, \\ \lambda x_3 = ab\pi_{12}\pi_{43}, & \lambda x_4 = a(c-b)\pi_{42}\pi_{43}. \end{array}$$

Therefore the two equations

$$\begin{aligned} (bx_2 - cx_3)\pi_{41} + (cx_3 - ax_1)\pi_{42} + (ax_1 - bx_2)\pi_{43} = 0, \\ a(b^2x_2 - c^2x_3)\pi_{41} + b(c^2x_3 - a^2x_1)\pi_{42} + c(a^2x_1 - b^2x_2)\pi_{43} = 0, \end{aligned}$$

(1) Eliminating x_1, x_2, x_3, x_4 among (38) we obtain

$\{a(b-c)\pi_{41} + b(c-a)\pi_{42} + c(a-b)\pi_{43}\}(bc\pi_{23}\pi_{41} + ca\pi_{31}\pi_{42} + ab\pi_{12}\pi_{43}) = 0.$

But the first factor does not vanish identically for (39).

combined with the condition of incidence may establish a one-to-one point-line correspondence, it is necessary and sufficient that (π_{ij}) should belong to the tetrahedral complex (40); and if the condition be satisfied, ∞^1 fields of W -force are obtained, as it is seen from § 9⁽¹⁾.

15. Now the general solution of the partial differential equation $F=0$ (31) for the W -force (34) takes the form

$$(42) \quad z = x^{\frac{c}{a}} \phi\left(\frac{y^a}{x^b}\right),$$

where ϕ denotes an arbitrary function, and the differential equation $G=0$ (32) becomes

$$(43) \quad a^2(b-c)^2x^2\left(\frac{\partial z}{\partial x}\right) - 2ab(b-c)(c-a)xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \\ + b^2(c-a)^2y^2\left(\frac{\partial z}{\partial y}\right)^2 + 2ca(a-b)(b-c)xz\frac{\partial z}{\partial x} \\ + 2bc(c-a)(a-b)yz\frac{\partial z}{\partial y} + c^2(a-b)^2z^2 = 0.$$

Since the tetrahedral complex (37) is invariant for the infinitesimal projective transformation

$$Uf \equiv ax\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y} + cz\frac{\partial f}{\partial z},$$

equation (43) is also invariant; hence it follows that there exist ∞^1 surfaces S (§ 10). In order to find these surfaces in a simple manner, if we put

$$\phi(\tau) = \text{const. } \tau^{-\frac{\mu}{av}},$$

and

$$(44) \quad a\lambda + b\mu + c\nu = 0,$$

(1) In order to obtain the lines of force we put

$$\frac{\pi_{12}}{xdy-ydx} = \frac{\pi_{23}}{ydz-zdy} = \frac{\pi_{31}}{zdx-xdz} = \frac{\pi_{41}}{dx} = \frac{\pi_{42}}{dy} = \frac{\pi_{43}}{dz}.$$

Then (38) becomes

$$(b-y-cz)dx + (cz-ax)dy + (ax-by)dz = 0, \\ a(b^2y-c^2z)dx + b(c^2z-a^2x)dy + c(a^2x-b^2y)dz = 0,$$

from which we have

$$\frac{dx}{ax} = \frac{dy}{by} = \frac{dz}{cz}.$$

This coincides with the result of § 12.

where λ, μ, ν are constants (which we have to determine), (42) becomes the W -surfaces

$$(45) \quad x^\lambda y^\mu z^\nu = k,$$

k being an arbitrary constant. In order that these surfaces satisfy the equation (43), it should be

$$(46) \quad a^2(b-c)^2\lambda^2 + b^2(c-a)^2\mu^2 + c^2(a-b)^2\nu^2 \\ - 2bc(c-a)(a-b)\mu\nu - 2ca(a-b)(b-c)\nu\lambda - 2ab(b-c)(c-a)\lambda\mu = 0.$$

Therefore when λ, μ, ν satisfy the two equations (44) and (46), the ∞^1 W -surfaces (45) belong to the surface $S^{(1)}$.

Hence in the field of W -force all the lines of force on any W -surface (45) are W -curves and they constitute a system of the asymptotic lines.

PART III. GEOMETRICAL TRANSFORMATIONS.

16. We have established a one-to-one correspondence between the lines of the tetrahedral complex

$$(47) \quad bc\pi_{23}\pi_{41} + ca\pi_{31}\pi_{42} + ab\pi_{12}\pi_{43} = 0$$

and the points of space, by the aid of the formulae⁽²⁾

$$(48) \quad \begin{aligned} \rho\pi_{41} &= ax_1x_4, & \rho\pi_{42} &= bx_2x_4, & \rho\pi_{43} &= cx_3x_4, \\ \rho\pi_{12} &= (b-a)x_1x_2, & \rho\pi_{23} &= (c-b)x_2x_3, & \rho\pi_{31} &= (a-c)x_3x_1; \end{aligned}$$

or

$$\begin{aligned} \lambda x_1 &= bc\pi_{23}\pi_{41}, & \lambda x_2 &= ca\pi_{23}\pi_{42}, \\ \lambda x_3 &= ab\pi_{23}\pi_{43}, & \lambda x_4 &= a(c-b)\pi_{42}\pi_{43}. \end{aligned}$$

This representation does not belong to that due to Prof. Noether by which a one-to-one correspondence can be established between the lines of a general quadratic complex and the points of space⁽³⁾.

(¹) Since (45) contains one arbitrary constant, it is the *unique* common complete integral of $F=0$, $G=0$ in the field of W -force.

(²) In Part III, x_1, x_2, x_3, x_4 are assumed to denote the *general* homogeneous point coordinates, and a, b, c to be *arbitrary* constants.

(³) Noether, Göttinger Nachrichten, (1869); Caporali, Mem. della R. Acad. dei Lincei, (3) 2 (1877-78); Jessop, Line complex (1903), p. 179.

Our transformation is also different from that of Weiler [Zeitschr. f. Math. u. Physik, 22 (1877), p. 261] which may be written in our notations:

$$\begin{aligned} \rho\pi_{41} &= a(b-c)x_1x_1, & \rho\pi_{42} &= b(a-c)x_2x_4, & \rho\pi_{43} &= c(a-b)x_3x_4, \\ \rho\pi_{12} &= a(c-b)x_1x_2, & \rho\pi_{23} &= a(c-b)x_2x_3, & \rho\pi_{31} &= a(b-c)x_3x_1. \end{aligned}$$

Let us consider the singular collineation of two spaces Σ and Σ' such that

$$\frac{1}{\rho}x'_1 = ax_1, \quad \frac{1}{\rho}x'_2 = bx_2, \quad \frac{1}{\rho}x'_3 = cx_3, \quad \frac{1}{\rho}x'_4 = 0 \cdot x_4.$$

If (π_{ij}) be the line joining any two corresponding points (x_i) and (x'_i) ,

$$\begin{aligned} \rho\pi_{41} &= ax_1x_4, & \rho\pi_{42} &= bx_2x_4, & \rho\pi_{43} &= cx_3x_4, \\ \rho\pi_{12} &= (b-a)x_1x_2, & \rho\pi_{23} &= (c-b)x_2x_3, & \rho\pi_{31} &= (a-c)x_3x_1. \end{aligned}$$

Thus we have obtained a method of formation of our representation (48).

To the lines of a congruence (2, 2):

$$\begin{aligned} l_1\pi_{41} + l_2\pi_{42} + l_3\pi_{43} + l_4\pi_{12} + l_5\pi_{23} + l_6\pi_{31} &= 0, \\ bc\pi_{23}\pi_{41} + ca\pi_{31}\pi_{42} + ab\pi_{12}\pi_{43} &= 0, \end{aligned}$$

correspond the points of the quadratic surface

$$al_1x_1x_4 + bl_2x_2x_4 + cl_3x_3x_4 + (b-a)l_4x_1x_2 + (c-b)l_5x_2x_3 + (a-c)l_6x_3x_1 = 0,$$

which passes through the four vertices of the fundamental tetrahedron.

To the condition that the two lines (π_{ij}) and (π'_{ij}) belonging to the tetrahedral complex may intersect, i. e.

$$\pi_{12}\pi'_{34} + \pi_{13}\pi'_{42} + \pi_{14}\pi'_{23} + \pi_{23}\pi'_{14} + \pi_{42}\pi'_{13} + \pi_{34}\pi'_{12} = 0,$$

corresponds the relation between (x_i) and (x'_i) such that

$$\begin{aligned} (49) \quad c(a-b)(x_1x_2x'_3x'_4 + x_3x_4x'_1x'_2) + b(c-a)(x_1x_3x'_2x'_4 + x_2x_4x'_1x'_3) \\ + a(b-c)(x_1x_4x'_2x'_3 + x_2x_3x'_1x'_4) = 0. \end{aligned}$$

17. Now consider the transformation between (x_i) and (x'_i) defined by the *aequatio directrix* (49) which is symmetrical with respect to (x_i) and (x'_i) .

To the point (x_i) corresponds the quadric

$$F_2(x') \equiv \Sigma c(a-b)(x_1x_2x'_3x'_4 + x_3x_4x'_1x'_2) = 0$$

in the current coordinates (x'_i) , which passes through the four vertices of the fundamental tetrahedron and the given point (x_i) . Conversely, to the point (x'_i) corresponds the quadric

$$F_2(x) \equiv \Sigma c(a-b)(x_1x_2x'_3x'_4 + x_3x_4x'_1x'_2) = 0$$

in the current coordinates (x_i) , which passes through the four vertices of the fundamental tetrahedron and the given point (x'_i) .

The quadric corresponding to any point on $F_2(x')=0$ passes through the point (x_i) , and conversely the quadric corresponding to any point on $F_2(x)=0$ passes through the point (x'_i) .

Now we prove that *this transformation is an involutory contact transformation*. In order to prove this, let us suppose that

$$x_1 \equiv a_1 x + b_1 y + c_1 z + d_1,$$

$$x_2 \equiv a_2 x + b_2 y + c_2 z + d_2,$$

$$x_3 \equiv a_3 x + b_3 y + c_3 z + d_3,$$

$$x_4 \equiv a_4 x + b_4 y + c_4 z + d_4.$$

We can choose the collineation between (x, y, z) and (ξ, η, ζ) in which the following relations are satisfied :

$$\lambda x_1 = \xi, \quad \lambda x_2 = \eta, \quad \lambda x_3 = \zeta, \quad \lambda x_4 = 1.$$

Then the aequatio directrix (49) becomes

$$(50) \quad \begin{aligned} \mathcal{Q} \equiv & c(a-b)(\xi \eta \zeta' + \zeta \xi' \eta') + b(c-a)(\xi \zeta' \eta' + \eta \xi' \zeta') \\ & + a(b-c)(\eta \zeta' \xi' + \xi \eta' \zeta') = 0. \end{aligned}$$

Hence the two equations

$$\frac{\partial \mathcal{Q}}{\partial \xi} + p \frac{\partial \mathcal{Q}}{\partial \zeta} = 0, \quad \frac{\partial \mathcal{Q}}{\partial \eta} + q \frac{\partial \mathcal{Q}}{\partial \zeta} = 0$$

become

$$(51) \quad \begin{cases} c(a-b)p\xi'\eta' + a(b-c)p\eta\xi' \\ \quad + \{b(c-a)p\xi + a(b-c)\zeta' + b(c-a)\zeta\}\eta' + c(a-b)\eta\zeta' = 0, \\ c(a-b)q\xi'\eta' + \{a(b-c)q\eta + b(c-a)\zeta' + a(b-c)\zeta\}\xi' \\ \quad + b(c-a)q\xi\eta' + c(a-b)\xi\zeta' = 0, \end{cases}$$

respectively. But since three equations (50), (51) are solvable with respect to ξ', η', ζ' , we can find the expressions of the form

$$(52) \quad \begin{cases} \xi' = X(\xi, \eta, \zeta, p, q), & \eta' = Y(\xi, \eta, \zeta, p, q), & \zeta' = Z(\xi, \eta, \zeta, p, q), \\ p' = P(\xi, \eta, \zeta, p, q), & q' = Q(\xi, \eta, \zeta, p, q), \end{cases}$$

where p', q' are defined by

$$\frac{\partial \mathcal{Q}}{\partial \xi'} + p' \frac{\partial \mathcal{Q}}{\partial \zeta'} = 0, \quad \frac{\partial \mathcal{Q}}{\partial \eta'} + q' \frac{\partial \mathcal{Q}}{\partial \zeta'} = 0.$$

Since \mathcal{Q} is symmetric with respect to $\xi, \xi'; \eta, \eta'$ and ζ, ζ' , we have from (52)

$$\begin{cases} \xi = X(\xi', \eta', \zeta', p', q'), & \eta = Y(\xi', \eta', \zeta', p', q'), & \zeta = Z(\xi', \eta', \zeta', p', q'), \\ p = P(\xi', \eta', \zeta', p', q'), & q = Q(\xi', \eta', \zeta', p', q'); \end{cases}$$

so that (ξ, η, ζ, p, q) and $(\xi', \eta', \zeta', p', q')$ are related by the involutory contact transformation which proves our theorem⁽¹⁾.

18. In the above contact transformation a straight line corresponds to a Kummer surface which can be transformed into the Kummer surface having the fundamental tetrahedron as the Rosenhain tetrahedron by a certain collineation leaving invariant the fundamental tetrahedron.

To prove this, let any point of a straight line be

$$\begin{aligned} x_1' &= a_1 \lambda + b_1 \mu, & x_2' &= a_2 \lambda + b_2 \mu, \\ x_3' &= a_3 \lambda + b_3 \mu, & x_4' &= a_4 \lambda + b_4 \mu, \end{aligned}$$

where λ, μ are homogeneous parameters and a, b are any constants; so that if we put

$$\rho_{ij} = a_i b_j - b_i a_j, \quad \rho_{ij} = -\rho_{ji}, \quad (i, j = 1, 2, 3, 4),$$

then

$$\rho_{12} \rho_{34} + \rho_{13} \rho_{42} + \rho_{14} \rho_{23} = 0.$$

Hence these six quantities (ρ_{ij}) are homogeneous coordinates of the given line.

Now all the points of this line correspond to a system of the quadrics

$$\begin{aligned} &\lambda^2 \Sigma c(a-b)(a_3 a_4 x_1 x_2 + a_1 a_2 x_3 x_4) \\ &+ \lambda \mu \Sigma c(a-b) \{ (a_3 b_4 + a_4 b_3) x_1 x_2 + (a_1 b_2 + a_2 b_1) x_3 x_4 \} \\ &+ \mu^2 \Sigma c(a-b)(b_3 b_4 x_1 x_2 + b_1 b_2 x_3 x_4) = 0. \end{aligned}$$

There are two quadrics of this system which pass through any given point, that is, the first characteristic of this system is 2. Hence the envelope of the quadrics must be a Kummer surface⁽²⁾.

In order to find the equation of this surface, let us put, for the sake of brevity,

$$\begin{aligned} A_{21} &= c(a-b) \rho_{21}, & A_{31} &= b(c-a) \rho_{31}, & A_{41} &= a(b-c) \rho_{41}, \\ A_{34} &= c(a-b) \rho_{34}, & A_{24} &= b(c-a) \rho_{24}, & A_{32} &= a(b-c) \rho_{32}, \\ A_{ij} &= -A_{ji} & & & (i, j = 1, 2, 3, 4). \end{aligned}$$

Then the envelope takes the form

$$\begin{aligned} &x_1^2 (A_{34}^2 x_3^2 + A_{24}^2 x_3^2 + A_{23}^2 x_4^2 + 2A_{34} A_{24} x_2 x_3 + 2A_{24} A_{23} x_3 x_4 + 2A_{34} A_{32} x_4 x_2) \\ (53) &+ 2x_1 \{ (A_{24} A_{14} x_3^2 + A_{32} A_{31} x_4^2) x_2 + (A_{34} A_{14} x_2^2 + A_{32} A_{12} x_4^2) x_3 \\ &\quad + (A_{34} A_{31} x_2^2 + A_{24} A_{21} x_3^2) x_4 + r x_2 x_3 x_4 \} \\ &+ (A_{41} x_2 x_3 + A_{31} x_2 x_4 + A_{21} x_3 x_4)^2 = 0, \end{aligned}$$

(1) Any collineation is of course a contact transformation.

(2) Zeuthen, Lehrbuch der abzählenden Methoden der Geometrie (1914), p. 154.

where

$$r = 2a^2(b-c)^2 \left[\frac{A_{13}A_{24}}{b^2(c-a)^2} + \frac{A_{12}A_{34}}{c^2(a-b)^2} \right] \\ + 2b^2(c-a)^2 \left[\frac{A_{12}A_{43}}{c^2(a-b)^2} + \frac{A_{23}A_{14}}{a^2(b-c)^2} \right] \\ + 2c^2(a-b)^2 \left[\frac{A_{32}A_{14}}{a^2(b-c)^2} + \frac{A_{13}A_{42}}{b^2(c-a)^2} \right].$$

Now we apply the collineation

$$\xi_1 = m_1 x_1, \quad \xi_2 = m_2 x_2, \quad \xi_3 = m_3 x_3, \quad \xi_4 = m_4 x_4, \quad (m_4 = 1),$$

such that

$$(54) \quad m_1 m_2 A_{12} = m_3 m_4 A_{34}, \quad m_2 m_3 A_{23} = m_1 m_4 A_{14}, \\ m_1 m_3 A_{13} = m_4 m_2 A_{42};$$

and therefore we may take

$$(55) \quad m_1 = \sqrt{\frac{A_{42}A_{34}}{A_{12}A_{13}}}, \quad m_2 = \sqrt{\frac{A_{14}A_{34}}{A_{12}A_{23}}}, \\ m_3 = \sqrt{\frac{A_{42}A_{14}}{A_{13}A_{23}}}, \quad m_4 = 1^{(1)}.$$

If we put

$$u = m_1 m_2 A_{12}, \quad v = m_1 m_3 A_{13}, \quad w = m_1 m_4 A_{14},$$

or

$$u = c(a-b)\rho_{34}\sqrt{\frac{\rho_{42}\rho_{14}}{\rho_{13}\rho_{23}}}, \quad v = b(c-a)\rho_{42}\sqrt{\frac{\rho_{34}\rho_{14}}{\rho_{12}\rho_{23}}}, \\ w = a(b-c)\rho_{14}\sqrt{\frac{\rho_{42}\rho_{34}}{\rho_{12}\rho_{13}}},$$

and

$$s = 2 \left(\frac{\rho_{14}\rho_{42}\rho_{34}}{\rho_{12}\rho_{13}\rho_{23}} \right)^2 \left[a^2(b-c)^2(\rho_{13}\rho_{24} + \rho_{12}\rho_{34}) + b^2(c-a)^2(\rho_{12}\rho_{43} + \rho_{14}\rho_{23}) \right. \\ \left. + c^2(a-b)^2(\rho_{14}\rho_{32} + \rho_{13}\rho_{42}) \right],$$

(¹) When, a, b, c and a_1, b_1, c_1, \dots are real, m_1, m_2, m_3 are all real in general. For, if this be not the case, since it is seen from (55) that $m_1 m_2 m_3$ is real, we may suppose for example, that

$$m_1 m_2 = \pm(M + iN), \quad m_3 = M - iN, \quad (M, N \text{ being real}).$$

Then from (54)

$$(M + iN)A_{12} = (M - iN)A_{34}, \quad \text{or} \quad -(M + iN)A_{12} = (M - iN)A_{34}.$$

Hence it must be

$$A_{12} = -A_{34} \quad \text{or} \quad A_{12} = A_{34},$$

which does not hold good in general.

equation (53) becomes

$$\begin{aligned} & \hat{\xi}_1^2(u^2\hat{\xi}_2^2+v^2\hat{\xi}_3^2+w^2\hat{\xi}_4^2-2vw\hat{\xi}_3\hat{\xi}_4-2wu\hat{\xi}_4\hat{\xi}_2-2uv\hat{\xi}_2\hat{\xi}_3) \\ & +2\hat{\xi}_1\{vw\hat{\xi}_2(\hat{\xi}_3^2-\hat{\xi}_4^2)+wu\hat{\xi}_3(\hat{\xi}_4^2-\hat{\xi}_2^2)+uv\hat{\xi}_4(\hat{\xi}_2^2-\hat{\xi}_3^2)+s\hat{\xi}_2\hat{\xi}_3\hat{\xi}_4\} \\ & +(u\hat{\xi}_3\hat{\xi}_4+v\hat{\xi}_4\hat{\xi}_2+w\hat{\xi}_2\hat{\xi}_3)^2=0, \end{aligned}$$

which is nothing but the Kummer surface referred to the Rosenhain tetrahedron⁽²⁾.

Ikeda near Ôsaka, September 1917.

(²) Hudson, Kummer's quartic surface (1905), p. 83.

On a Solution of the Wave-Equation,

by

II. BATEMAN, Pasadena, California, U. S. A.

1. Let us write

$$S(x, y, z, t, \tau) \equiv [x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \zeta(\tau)]^2 - c^2(t - \tau)^2,$$

where c is a constant and $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$ are arbitrary functions of τ , then if $f(\tau)$ is another arbitrary function of τ subject to certain limitations, the definite integral

$$V = \int_{\tau_2}^{\tau_1} \frac{f(\tau) d\tau}{S(x, y, z, t, \tau)}$$

satisfies the wave-equation

$$\square V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = 0,$$

provided τ_2 and τ_1 are values of τ which satisfy the partial differential equations

$$\left(\frac{\partial \tau}{\partial x}\right)^2 + \left(\frac{\partial \tau}{\partial y}\right)^2 + \left(\frac{\partial \tau}{\partial z}\right)^2 = \frac{1}{c^2} \left(\frac{\partial \tau}{\partial t}\right)^2, \quad (1)$$

$$S(x, y, z, t, \tau) \square \tau = 4 \left\{ [x - \xi(\tau)] \frac{\partial \tau}{\partial x} + [y - \eta(\tau)] \frac{\partial \tau}{\partial y} + [z - \zeta(\tau)] \frac{\partial \tau}{\partial z} + (t - \tau) \frac{\partial \tau}{\partial t} \right\}. \quad (2)$$

A particular solution of these equations has already been found⁽¹⁾ in the case when $\xi(\tau)$, $\eta(\tau)$ and $\zeta(\tau)$ are constants. This solution will now be generalised as follows:—

Let a point P whose co-ordinates at time θ are α, β, γ move along a curve Γ with a velocity which is always less than c and let τ be defined in terms of x, y, z and t by means of the equations

$$[x - \alpha(\theta)]^2 + [y - \beta(\theta)]^2 + [z - \gamma(\theta)]^2 = c^2(t - \theta)^2, \quad (3)$$

$$[\alpha(\theta) - \xi(\tau)]^2 + [\beta(\theta) - \eta(\tau)]^2 + [\gamma(\theta) - \zeta(\tau)]^2 = c^2(\theta - \tau)^2. \quad (4)$$

(¹) See the author's Electrical and Optical Wave Motion, p. 30.

If we assume that the point Q , whose co-ordinates at time τ are $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$, moves in a continuous manner with a velocity which is always less than c and if we introduce the inequalities $\theta < t$, $\tau < \theta$, then θ and τ are defined uniquely⁽¹⁾ by the above equations. We can verify that the function τ so defined satisfies our partial differential equations by noticing that

$$x - a = M \frac{\partial \theta}{\partial x}, \quad \frac{\partial \tau}{\partial x} = \frac{\partial \tau}{\partial \theta} \frac{\partial \theta}{\partial x}, \quad \frac{\partial^2 \tau}{\partial x^2} = \frac{\partial \tau}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \tau}{\partial \theta^2} \left(\frac{\partial \theta}{\partial x} \right)^2,$$

where

$$M = (x - a) a'(\theta) + (y - \beta) \beta'(\theta) + (z - \gamma) \gamma'(\theta) - c^2(t - \theta).$$

Equation (1) is evidently satisfied and we easily find that $\square \tau = \frac{\partial \tau}{\partial \theta} \square \theta$.

Now I have shown elsewhere⁽²⁾ that $\square \theta = \frac{2}{M}$, hence it is easy to see that equation (2) is satisfied if

$$S(x, y, z, t, \tau) = 2 \{ [x - \xi(\tau)] [x - a(\theta)] + [y - \eta(\tau)] [y - \beta(\theta)] \\ + [z - \zeta(\tau)] [z - \gamma(\theta)] - c^2(t - \tau)(t - \theta) \}.$$

This equation may be obtained, however, by subtracting (4) from (3); consequently the function τ defined by equations (3) and (4) is a solution of equations (1) and (2). If τ_1 is defined with the aid of a point P_1 which moves along a curve Γ_1 and τ_2 with the aid of a point P_2 which moves along a curve Γ_2 , a wave-function is obtained which possesses the following characteristics:—

Let particles be fired out from the different positions of Q so that they will travel along straight lines with the velocity c and hit either P_1 or P_2 , then if a particle is regarded simply as a moving point it will become a moving point singularity of the function V after it has hit either P_1 or P_2 . This is evident because $S(x, y, z, t, \tau)$ is zero for such a particle after it has hit either P_1 or P_2 .

Let us now consider the case when the particles which are fired out from Q hit both P_1 and P_2 and the time which elapses between the two impacts is infinitesimal.

We may evidently write

$$\begin{aligned} a_1(\theta_1) &= \xi(\tau_1) + cl(\tau_1)\chi(\tau_1), & \gamma_1(\theta_1) &= \zeta(\tau_1) + cn(\tau_1)\chi(\tau_1), \\ \beta_1(\theta_1) &= \eta(\tau_1) + cm(\tau_1)\chi(\tau_1), & \theta_1 &= \tau_1 + \chi(\tau_1), \end{aligned}$$

(¹) A. Liénard, L'éclairage électrique, Vol. 16. (1898), p. 5.

(²) Annals of Mathematics, March (1914)—Dec. (1912).

$$\begin{aligned} a_2(\theta_2) &= \xi(\tau_2) + el(\tau_2)\zeta'(\tau_2), & \gamma_2(\theta_2) &= \zeta'(\tau_2) + en(\tau_2)\zeta'(\tau_2), \\ \beta_2(\theta_2) &= \eta(\tau_2) + em(\tau_2)\zeta'(\tau_2), & \theta_2 &= \tau_2 + \phi'(\tau_2), \end{aligned}$$

where $l(\tau), m(\tau), n(\tau)$ are the direction cosines of a line. The condition that the interval between the two impacts is to be small may then be satisfied by writing

$$\phi'(\tau) = \chi(\tau) + \varepsilon\phi(\tau),$$

where $\phi(\tau)$ is finite and ε is a small quantity whose square may be neglected. Subtracting the two equations

$$\begin{aligned} [x - a_1(\theta_1)]^2 + [y - \beta_1(\theta_1)]^2 + [z - \gamma_1(\theta_1)]^2 &= c^2(t - \theta_1)^2, \\ [x - a_2(\theta_2)]^2 + [y - \beta_2(\theta_2)]^2 + [z - \gamma_2(\theta_2)]^2 &= c^2(t - \theta_2)^2, \end{aligned}$$

and neglecting squares and higher powers of $\tau_2 - \tau_1$ and $\theta_2 - \theta_1$, we find that

$$\begin{aligned} M_1(\tau_2 - \tau_1) \frac{d\theta_1}{d\tau_1} + \varepsilon\phi(\tau_1) \{ l(\tau_1)[x - a_1(\theta_1)] + m(\tau_1)[y - \beta_1(\theta_1)] \\ + n(\tau_1)[z - \gamma_1(\theta_1)] - c(t - \theta_1) \} = 0. \end{aligned}$$

In this equation the coefficient of $\varepsilon\phi(\tau_1)$ may be replaced by the equivalent expression

$$-\frac{1}{2\chi(\tau_1)} S(x, y, z, t, \tau_1),$$

which is obtained by using the first of the two preceding equations. It is now easy to see that the definite integral

$$V = \int_{\tau_2}^{\tau_1} \frac{f(\tau) d\tau}{S(x, y, z, t, \tau)} = \frac{(\tau_1 - \tau_2)f(\tau_1)}{S(x, y, z, t, \tau_1)}$$

reduces to

$$-\frac{\varepsilon}{2M_1} \frac{f(\tau_1)\phi(\tau_1)}{\chi(\tau_1)} \frac{d\tau_1}{d\theta_1}$$

and so is of the form

$$\frac{1}{M_1} F(\theta_1).$$

Now this is just the type of solution which occurs in the specification of the electromagnetic potentials of a moving point charge of electricity associated with the point P_1 and since for this solution P_1 is the only moving singularity it appears that there is a cancelling of the singularities produced by P_1 and P_2 after the second point has been hit by the

particles from Q ; it is only on the infinitesimal interval between P_1 and P_2 that the singularities appear. In the case when P_1, P_2 and Q are stationary and $f(\tau)=1$ it is easy to see that

$$V = \frac{1}{2Rc} \left(\log \frac{R+r_2+\rho_2}{R-r_2-\rho_2} - \log \frac{R+r_1+\rho_1}{R-r_1-\rho_1} \right),$$

where $\rho_1 = P_1Q$, $\rho_2 = P_2Q$ and R, r_1, r_2 are the distances of the point (x, y, z) from Q, P_1 and P_2 respectively. When P_1, P_2 and Q are collinear and $P_1P_2 = \delta$ is infinitesimal, the above expression reduces to

$$-\frac{\delta}{2cr_1\rho_1},$$

which is the electrostatic potential of a point charge of electricity at P_1 . To get a charge which is not small we must take f to be a large constant instead of unity. The solution of the wave-equation which is represented by our definite integral is evidently of some interest in connection with the theory of electricity which has been developed elsewhere⁽¹⁾.

2. It should be noticed that a rather peculiar phenomenon occurs when the points P and Q describe straight lines with the velocity of light c instead of moving along curves with a smaller velocity. Let us call a particle which moves along a straight line with the velocity c a light-particle, then it appears that P cannot be hit by a light-particle from Q until after it has passed a certain point P_0 on its path and the light-particles which are fired out from Q , after Q has passed a certain point Q_0 on its path, cannot hit P . Likewise Q cannot be hit by a light-particle fired out from P until it has passed the point Q_0 , while the light-particles fired out from P , after it has passed the point P_0 , cannot hit Q . To prove these theorems⁽²⁾ let us take the line of shortest distance between the rectilinear paths of P and Q as axis of z , then the motions of P and Q may be specified as follows:—

$$\begin{array}{lll} P & x = c(t + \lambda) \cos \alpha, & y = c(t + \lambda) \sin \alpha, & z = kc, \\ Q & x = c(t + \mu) \cos \alpha, & y = -c(t + \mu) \sin \alpha, & z = -kc. \end{array}$$

A light-particle is at P and Q at times t_1 and t_2 respectively, if

$$(t_1 - t_2 + \lambda - \mu)^2 \cos^2 \alpha + (t_1 + t_2 + \lambda + \mu)^2 \sin^2 \alpha + 4k^2 = (t_1 - t_2)^2.$$

(¹) Washington Acad. of Sciences, April (1917). A fuller account of the mathematical theory has been offered for publication in the Proceedings of the London Mathematical Society.

(²) These theorems are probably well known, but they are mentioned here to emphasize the desirability of a further study of the geometry of light-particles.

Writing this equation in the form

$$[t_1 - t_2][2(\lambda - \mu) \cos^2 \alpha + 2(2t_2 + \lambda + \mu) \sin^2 \alpha] \\ + (\lambda - \mu)^2 \cos^2 \alpha + (\lambda + \mu + 2t_2)^2 \sin^2 \alpha + 4k^2 = 0,$$

we see that $t_2 - t_1$ is positive or negative according as

$$2t_2 + \lambda + \mu + (\lambda - \mu) \cot^2 \alpha$$

is greater or less than zero. By putting this quantity equal to zero we may determine the position of Q_0 . In a similar way we find that the position of P_0 is determined by the equation

$$2t_1 + \lambda + \mu + (\mu - \lambda) \cot^2 \alpha = 0.$$

Since $t_1 + \lambda = (\lambda - \mu) \operatorname{cosec}^2 \alpha$ and $t_2 + \mu = (\mu - \lambda) \operatorname{cosec}^2 \alpha$, it appears that the points P_0 and Q_0 are equidistant from the line of shortest distance between the paths of P and Q .

3. It is easy to see that a solution of the n -dimensional wave-equation

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \dots + \frac{\partial^2 V}{\partial x_n^2} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}$$

may be obtained by using a definite integral similar to the above but with $S^{\frac{n-1}{2}}$ in place of S . The simplification which occurs when P_1 and P_2 are hit at consecutive instants by the same light-particle from Q is peculiar, however, to the case in which $n=3$.

Modified Extensions of Theorems of Bayes and Laplace,

by

MAGOICHIRO WATANABE, Tôkyô.

It is well known as immediate consequences from the theorem of Bernoulli and that of Bayes respectively that

(i) *If the probability that an event will occur on every single trial be p and the number of the happening of the event in the course of n independent trials be s , then the probability that the difference $\left| \frac{s}{n} - p \right|$ is smaller than an arbitrarily chosen positive number ε approaches 1 as limit when n tends to become infinite.*

(ii) *If p denote the unknown probability of an event and in n independent trials the event has happened just s times, then the probability that the difference $\left| p - \frac{s}{n} \right|$ is smaller than an arbitrarily chosen positive number ε approaches 1 as limit when n tends to become infinite.*

Of these results the former was extended by me by means of Tchebycheff's criterion to some classes of dependent trials⁽¹⁾. Whether similar extensions can be made of the latter is a question still unsolved. No result has been obtained, so far as I can, for extensions in its original form even in the simplest case where any two trials r_1^{th} and r_2^{th} are independent to each other if $|r_1 - r_2|$ exceeds a definite finite number which is independent of p . It is easy, however, to derive a theorem which can be looked upon as an extension of it in a slightly modified form. It runs as follows:—

If p denote the unknown probability of an event in a single trial a priori and the number s of the happening of the event in n trials dependent or independent is known to lie between two numbers $n(P - \lambda)$ and $n(P + \lambda)$ ⁽²⁾, where P and λ are definite positive numbers such that $\lambda < P < 1 - \lambda$, then the probability that the inequality $|p - P| \leq \lambda$ holds approaches 1 as limit when

⁽¹⁾ This Journal, Vol. 12. (1917) p. 24-42.

⁽²⁾ λ may be taken as small as we please.

n becomes infinite provided that certain conditions be satisfied⁽¹⁾.

The deduction of the theorem and its further extension to some directions analogous to that treated by Laplace constitute the main subject of the present note.

1. Let p denote the unknown probability of an event in every single trial, it being assumed that the probability in every single trial be the same for all the trials *a priori* and let $F_n(p; s)$ denote the probability of the happening just s times of an event in n trials, *dependent or independent*. Now if the number s of the happening of the event is known to lie between two numbers $n(P-\lambda)$ and $n(P+\lambda)$ where P and λ are definite positive numbers such that $\lambda < P < 1-\lambda$, then the probability π_n that the inequality

$$|p-P| \leq \lambda$$

holds, is given by

$$\pi_n = \frac{\alpha_n}{\beta_n} \quad (2)$$

provided that β_n does not vanish, where α_n and β_n are some intermediate numbers between the upper and the lower integrals of

$$\int_{P-\lambda}^{P+\lambda} \Sigma' F_n(p; s) dp \quad \text{and} \quad \int_0^1 \Sigma' F_n(p; s) dp, \quad (2)$$

respectively, Σ' denoting the summation with respect to s which satisfies

$$n(P-\lambda) \leq s \leq n(P+\lambda).$$

Here the upper and the lower integrals in (2) may be taken in the sense of Riemann as well as in that of Lebesgue⁽³⁾, and as similar discussions can be applied to both cases we shall take the former to show that π_n tends to unity when $n=\infty$ if certain conditions be satisfied.

(1) It can be shown in an almost similar manner to that given hereafter that when the same conditions be satisfied the inverse theorem also holds, i.e.

If the probability of an event in every single trial is constant and is known to lie between two numbers $(P-\lambda)$ and $(P+\lambda)$, then the probability that the number of the happening of the event in n trials lies between $n(P-\lambda)$ and $n(P+\lambda)$ approaches 1 as limit when n becomes infinite.

But this theorem will be of little value just now where we have a direct extension of Bernoulli's result.

(2) We tacitly assume the principle of equal distribution of ignorance.

(3) The upper and the lower integrals of a function $f(x)$ in the interval (a, b) in the sense of Lebesgue are defined to be the exterior and the interior measures (of Borel) of a two dimensional set of points (x, y) determined by $a \leq x \leq b$, $0 \leq y \leq f(x)$.

First consider the expression a_n . Since Σ' is the summation of s which satisfies

$$P-p-\lambda \leq \frac{s}{n} - p \leq P-p+\lambda, \quad (3)$$

we have, if $P-\lambda < p \leq P$,

$$\Sigma' F_n(p; s) \geq \sum_{\left| \frac{s}{n} - p \right| \leq |P-p-\lambda|} F_n(p; s),$$

for, in this case, the three relations

$$P-p-\lambda < 0, \quad P-p+\lambda > 0,$$

$$|P-p-\lambda| \leq P-p+\lambda$$

exist.

Accordingly by Tchebycheff's criterion⁽¹⁾

$$\Sigma' F_n(p; s) > 1 - \frac{\epsilon_n^2(p)}{n^2(P-p-\lambda)^2}, \quad (4)$$

where $\epsilon_n(p)$ denotes the mean error of the frequency number s defined by

$$\epsilon_n(p) = \sum_{s=0}^n \{s - np\}^2 F_n(p; s).$$

Similarly, if $P < p < P+\lambda$,

$$\begin{aligned} \Sigma' F_n(p; s) &\geq \sum_{\left| \frac{s}{n} - p \right| \leq (P-p+\lambda)} F_n(p; s) \\ &> 1 - \frac{\epsilon_n^2(p)}{n^2(P-p+\lambda)^2}, \end{aligned}$$

and therefore, by taking two arbitrary positive numbers ϵ and ϵ' each smaller than λ ,

$$\begin{aligned} &\int_{P-\lambda+\epsilon}^{P+\lambda-\epsilon'} \Sigma' F_n(p; s) dp \\ &\geq \int_{P-\lambda+\epsilon}^P \left\{ 1 - \frac{\epsilon_n^2(p)}{n^2(P-p-\lambda)^2} \right\} dp + \int_P^{P+\lambda-\epsilon'} \left\{ 1 - \frac{\epsilon_n^2(p)}{n^2(P-p+\lambda)^2} \right\} dp, \end{aligned}$$

(1) L. c. p. 24.

$$\geq 2\lambda - \varepsilon - \varepsilon' - \left\{ \int_{P-\lambda+\varepsilon}^{\overline{P}} \frac{\varepsilon_n^2(p) dp}{n^2(P-p-\lambda)^2} + \int_P^{\overline{P+\lambda-\varepsilon'}} \frac{\varepsilon_n^2(p) dp}{n^2(P-p+\lambda)^2} \right\}^{(1)}.$$

Since $\frac{\varepsilon_n(p)}{n}$ is positive and limited⁽²⁾ for all n and p ($0 \leq p \leq 1$) it follows that

$$\lim_{n \rightarrow \infty} \int_{\underline{P-\lambda+\varepsilon}}^{\overline{P+\lambda-\varepsilon'}} \Sigma' F_n(p; s) dp \geq 2\lambda - \varepsilon - \varepsilon',$$

if the condition

$$\lim_{n \rightarrow \infty} \int_0^1 \left\{ \frac{\varepsilon_n(p)}{n} \right\}^2 dp \rightarrow 0 \quad (5)$$

be satisfied. Since again $\Sigma' F_n(p; s)$ is limited⁽³⁾ for all n and p ($0 \leq p \leq 1$) and $\varepsilon, \varepsilon'$ are arbitrarily chosen positive numbers we shall have

$$\lim_{n \rightarrow \infty} \int_{\underline{P-\lambda}}^{\overline{P+\lambda}} \Sigma' F_n(p; s) dp \geq 2\lambda.$$

On the other hand, it is easily seen that

$$2\lambda \leq \int_{\underline{P-\lambda}}^{\overline{P+\lambda}} \Sigma' F_n(p; s) dp \leq \int_{\underline{P-\lambda}}^{\overline{P+\lambda}} \Sigma' F_n(p; s) dp,$$

and then we must have necessarily

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \int_{\underline{P-\lambda}}^{\overline{P+\lambda}} \Sigma' F_n(p; s) dp = 2\lambda,$$

provided that the condition (5) is satisfied.

Next consider the expression β_n . We have again, if $0 < p < P - \lambda$, by taking a positive number ε less than $P - \lambda$,

(1) \int and \int are the symbols denoting upper and lower integrals respectively.

It can easily be shown that, if $f_1(x)$ be integrable in (a, b) then

$$\int_a^b \{f_1(x) + f_2(x)\} dx = \int_a^b f_1(x) dx + \int_a^b f_2(x) dx = \int_a^b f_1(x) dx - \int_a^b \{-f_2(x)\} dx$$

(2) $\frac{\varepsilon_n(p)}{n}$ lies between p and $1-p$.

(3) $\Sigma' F_n(p; s)$ lies between 0 and 1.

$$\left| \frac{s}{n} - p \right| \leq P - p - \lambda - \epsilon \quad \Sigma F_n(p; s) > 1 - \frac{\epsilon_n^2(p)}{n^2(P - p - \lambda - \epsilon)^2},$$

for values of p satisfying $0 < p < P - \lambda - \epsilon$ so that

$$\begin{aligned} \Sigma' F_n(p; s) &\leq \Sigma F_n(p; s) \\ &\left| \frac{s}{n} - p \right| \leq P - p - \lambda \\ &< \frac{\epsilon_n^2(p)}{n^2(P - p - \lambda - \epsilon)^2}, \end{aligned}$$

while, when $p=0$, $\Sigma' F_n(p; s)$ vanishes. As $\Sigma' F_n(p; s)$ is limited even in the vicinity of $p=P-\lambda-\epsilon$ for all n , it will follow that the upper integral

$$\int_0^{P-\lambda-\epsilon} \Sigma' F_n(p; s) dp$$

and consequently also the upper integral

$$\int_0^{P-\lambda} \Sigma' F_n(p; s) dp$$

will vanish in the limit when n becomes infinite if the condition (5) be satisfied.

Similarly the upper integral

$$\int_{P+\lambda}^1 \Sigma' F_n(p; s) dp$$

vanishes when n becomes infinite. Since the corresponding two lower integrals can never be negative we shall have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 \Sigma' F_n(p; s) dp &= \lim_{n \rightarrow \infty} \int_{P-\lambda}^{P+\lambda} \Sigma' F_n(p; s) dp \\ &= 2\lambda \end{aligned}$$

which gives $\lim_{n \rightarrow \infty} \beta_n = 2\lambda$.

Thus we have obtained under the condition (5)

$$\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 2\lambda \neq 0,$$

and accordingly

$$\lim_{n \rightarrow \infty} \pi_n \rightarrow 1, \quad (6)$$

which is to be proved.

In case when the integral $\int_0^1 \epsilon_n^2(p) dp$ exists in the sense of Riemann the condition (5) is replaced by

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{\epsilon_n^2(p)}{n^2} dp \rightarrow 0. \quad (7)$$

Since the above reasoning applies equally when the integrals are taken in the sense of Lebesgue we have the theorem:

If p denote the unknown probability of an event in every single trial a priori and the number s of the happening of the event in n trials depending in such a manner that one of the following conditions is satisfied, is known to lie between two numbers $n(P-\lambda)$ and $n(P+\lambda)$, where P and λ are definite positive numbers such that $\lambda < P < 1-\lambda$, then the probability that the inequality $|p-P| \leq \lambda$ holds approaches 1 as limit when n becomes infinite:

$$(a) \quad \lim_{n \rightarrow \infty} \frac{1}{n^2} \int_0^1 \epsilon_n^2(p) dp \rightarrow 0, \quad (5)$$

the upper integral being taken in the sense of Riemann or of Lebesgue.

(b) The integral $\int_0^1 \epsilon_n^2(p) dp$ exists in the sense of Riemann or of Lebesgue and

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \int_0^1 \epsilon_n^2(p) dp \rightarrow 0. \quad (7)$$

2. Simple sufficient conditions which are derivable from the foregoing will now be considered.

(i) The condition (5) is satisfied if $\frac{\epsilon_n(p)}{n}$ converges uniformly to zero for all p between 0 and 1 when n becomes infinite; or more generally, if the set A complementary⁽¹⁾ to the set of points p of uniform convergence of $\frac{\epsilon_n(p)}{n}$ to the value zero is of the measure zero⁽²⁾.

For if the set A which is necessarily closed has zero measure then all the points belonging to it can be enclosed in a finite number of

(¹) The set A is identical with the set of points of non-uniform convergence in the case when $\frac{\epsilon_n(p)}{n} \rightarrow 0$ for all p but not uniformly.

(²) According to Borel's definition.

intervals the sum of whose lengths is less than an arbitrarily chosen positive number ε . The remaining part of the interval $(0, 1)$ consists of a finite set of intervals within each of which $\frac{\varepsilon_n(p)}{n}$ converges uniformly to zero. Since $\frac{\varepsilon_n(p)}{n}$ is limited for all n and p ($0 \leq p \leq 1$) it follows that

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{\varepsilon_n^2(p)}{n^2} dp \rightarrow 0.$$

(ii) In case when the Riemann integral $\int_0^1 \varepsilon_n^2(p) dp$ exists, the condition (7) is satisfied if

$$\lim_{n \rightarrow \infty} \frac{\varepsilon_n(p)}{n} \rightarrow 0,$$

the convergence being not necessarily uniform; or more generally, if the set A of points p of non-convergence of $\frac{\varepsilon_n(p)}{n}$ to zero, when the set is closed by the addition of its limiting points, is of the measure zero.

For if $\lim_{n \rightarrow \infty} \frac{\varepsilon_n(p)}{n} = \phi(p)$, then $\phi(p)$ vanishes except at points of a set A whose measure is zero and therefore is an integrable null-function in the sense of Riemann if the set A , when closed by the addition of its limiting points, is of the measure zero so that the Riemann integral

$$\int_0^1 \phi(p) dp$$

vanishes. Since the functions $\frac{\varepsilon_n(p)}{n}$ and $\phi(p)$ are limited for all n and p ($0 \leq p \leq 1$) and also integrable we have

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{\varepsilon_n^2(p)}{n^2} dp = \int_0^1 \phi^2(p) dp^{(1)},$$

and therefore

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{\varepsilon_n(p)}{n^2} dp \rightarrow 0.$$

(iii) In case when the Lebesgue integral $\int_0^1 \varepsilon_n^2(p) dp$ exists, the condition

(1) Hobson: Theory of functions of a real variable, p. 539, § 383.

(7) is satisfied if

$$\lim_{n \rightarrow \infty} \frac{\epsilon_n(p)}{n} \rightarrow 0,$$

the convergence being not necessarily uniform: or more generally, if the set A of points of non-convergence of $\frac{\epsilon_n(p)}{n}$ to zero is of the measure zero⁽¹⁾.

For, in this case, the function $\phi(p)$ defined in the above is an integrable null-function in the sense of Lebesgue and hence

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{\epsilon_n^2(p)}{n^2} dp = \int_0^1 \phi^2(p) dp = 0 \quad (2).$$

3. The condition found in the preceding article may also be expressed in a somewhat different form.

I have shown in my previous paper⁽³⁾ that if $f_{m,r}(p)$ denote the probability of the concurrence of the event in two trials m^{th} and $(m+r)^{\text{th}}$, the occurrence or non-occurrence in the other trials being disregarded, then

$$\epsilon_n^2(p) = npq + 2V_n, \quad (8)$$

where

$$q = 1 - p, \quad V_n = \sum \{ f_{m,r}(p) - p^2 \},$$

the summation being extended over

$$r = 1, 2, \dots, n - m; \quad m = 1, 2, \dots, n - 1,$$

and by aid of this formula that if

$$\lim_{r \rightarrow \infty} f_{m,r}(p) \rightarrow p^2 \quad (4),$$

(1) This case includes the former case.

(2) Hobson: l. c. p. 512, § 384.

In this case the derived set of A may have a measure greater than zero. For example let us define $F_n(p; s)$ as follows:

if p be irrational all the trials are independent and if p be rational

$$F_n(p; n) = p, \quad F_n(p; 0) = 1 - p, \quad F_n(p; s) = 0 \quad (1 \leq s \leq n - 1).$$

That such a case may really occur can easily be seen. Then we have

$$\epsilon_n^2(p) = np(1 - p) \quad \text{if } p \text{ be irrational,}$$

$$\epsilon_n^2(p) = n^2 p(1 - p) \quad \text{if } p \text{ be rational,}$$

so that the set A consists of all rational points while A' of all numbers between 0 and 1. Still in such a case the condition (iii) is satisfied and hence our theorem follows at once.

(3) L. c., p. 30.

(4) A more general case may similarly be treated. But as it rarely occurs we shall not enter into the discussion of such a case.

uniformly for all values of m , then

$$\lim_{n \rightarrow \infty} \frac{\varepsilon_n^2(p)}{n^2} \rightarrow 0.$$

It can be shown, in quite a similar manner, that if $f_{m,r}(p)$ converges uniformly within a domain of p and for all values of m to the value p^2 , then $\frac{\varepsilon_n(p)}{n}$ converges also uniformly to the value zero within the same domain of p .

Accordingly we have the result:

The theorem enunciated in § 1 holds in one of the following cases:

(a) When

$$\lim_{r \rightarrow \infty} f_{m,r}(p) \rightarrow p^2,$$

uniformly for all p ($0 \leq p \leq 1$) and for all integers m ; or more generally, when the set complementary to the set B of points which is such that in the vicinity of the point belonging to the set B and for all integers m the function $f_{m,r}(p)$ converges uniformly to p^2 when $r \rightarrow \infty$, is of the measure zero.

(b) When the Riemann integrals

$$\int_0^1 f_{m,r}(p) dp \quad \begin{array}{l} m=1, 2, 3, \dots; \\ r=1, 2, 3, \dots \end{array}$$

exist and also

$$\lim_{r \rightarrow \infty} f_{m,r}(p) \rightarrow p^2,$$

for all p between 0 and 1 or except at points of a set which, when closed by the addition of its limiting points, is of the measure zero, the convergence being uniform with respect to m but not necessarily with respect to p .

(c) When the Lebesgue integrals

$$\int_0^1 f_{m,r}(p) dp \quad \begin{array}{l} m=1, 2, 3, \dots \\ r=1, 2, 3, \dots \end{array}$$

exist and also

$$\lim_{r \rightarrow \infty} f_{m,r}(p) \rightarrow p^2,$$

for all p between 0 and 1 or except at points of a set of zero measure, which, when closed by the addition of its limiting points, may be of any measure, the convergence being uniform with respect to m .

In the case when $f_{m,r}(p)$ is independent of m , the uniform convergence with respect to m is always satisfied by

$$f_r(p) \rightarrow p^2,$$

where

$$f_{m,r}(p) = f_r(p).$$

As an example let us consider the case where any two trials r_1^{th} and r_2^{th} are independent to each other if $|r_1 - r_2| \geq k$, k being a definite number independent of p . In this case

$$f_{m,r}(p) = p^2 \quad \text{if } r \geq k$$

and therefore by the criterion (a) we have the validity of our theorem for the case.

As another example⁽¹⁾ we take the case of a set of games played between two persons A and B which can not be drawn and in which A 's chance of winning in the first game is p and in every following game it is

$$\bar{p} = \frac{p}{k + (1-k)p} \quad \text{or} \quad \bar{q} = \frac{kp}{k + (1-k)p},$$

k being a positive constant less than 1, according as he wins in the game before or not. In this case it can easily be seen that A 's chance of winning in every single game is p *a priori*. For

$$p\bar{p} + q\bar{q} = p, \quad \text{where } q = 1 - p.$$

Again we have

$$f_r(p) = (\bar{p} - \bar{q})f_{r-1}(p) + p\bar{q},$$

a recursion formula for $f_r(p)$ from which we obtain

$$\begin{aligned} f_r(p) - p^2 &= (\bar{p} - \bar{q})\{f_{r-1}(p) - p^2\}, \\ &= (\bar{p} - \bar{q})^{r-1}p(p-p), \\ &= \left\{ \frac{(1-k)p}{k + (1-k)p} \right\}^{r-1} \left(\frac{1}{k + (1-k)p} - 1 \right) p^2. \quad (9) \end{aligned}$$

Accordingly by the criterion (a) or (b) we get the validity of our theorem for the case.

4. We have shown in the preceding articles that

$$\lim_{n \rightarrow \infty} \pi_n \rightarrow 1,$$

when certain conditions are satisfied. We shall now extend the theorem in the following manner.

(1) This example was taken from my previous paper.

Let p_1, p_2, \dots, p_l denote the unknown probabilities of the exclusive events w_1, w_2, \dots, w_l such that

$$p_1 + p_2 + \dots + p_l = 1, \quad (10)$$

and let s_1, s_2, \dots, s_l be the numbers of the happening of the events w_1, w_2, \dots, w_l in n trials dependent or independent respectively, so that

$$s_1 + s_2 + \dots + s_l = n, \quad (11)$$

then if it is known that

$$P_1 - \lambda_1 \leq \frac{s_1}{n} \leq P_1 + \lambda_1, \quad P_2 - \lambda_2 \leq \frac{s_2}{n} \leq P_2 + \lambda_2, \dots, \\ P_{l-1} - \lambda_{l-1} \leq \frac{s_{l-1}}{n} \leq P_{l-1} + \lambda_{l-1}, \quad (12)$$

where P_1, P_2, \dots, P_{l-1} are positive constants whose sum is smaller than 1 and $\lambda_1, \lambda_2, \dots, \lambda_{l-1}$ are positive numbers subject to the conditions

$$\lambda_s < P_s, \quad (s=1, 2, \dots, l-1)$$

$$\sum_{s=1}^{l-1} P_s < 1 - \sum_{s=1}^{l-1} \lambda_s,$$

the probability π_n that a system of the inequalities

$$P_s - \lambda_s \leq p_s \leq P_s + \lambda_s \quad (s=1, 2, \dots, l-1)$$

holds approaches 1 as limit when n becomes infinite provided that certain conditions are satisfied.

As the general case may similarly be treated, we shall here consider the case when $l=3$.

If p_1, p_2, p_3 be the probabilities of three exclusive events w_1, w_2, w_3 such that

$$p_1 + p_2 + p_3 = 1$$

and if $F_n(p_1; s_1)$ be the probability of the happening just s_1 times of the event w_1 and $F_n'(p_2; s_2)$ be that of the happening just s_2 times of the event w_2 , the occurrence or non-occurrence of the other events being disregarded in each case, then it can be shown that the probability that the inequalities

$$\left| \frac{s_1}{n} - p_1 \right| \leq \lambda_1 \quad \text{and} \quad \left| \frac{s_2}{n} - p_2 \right| \leq \lambda_2$$

hold is greater than

$$1 - \left\{ \frac{\varepsilon_n^2(p_1)}{n^2 \lambda_1^2} + \frac{\varepsilon_n'^2(p_2)}{n^2 \lambda_2^2} \right\}^{(1)}, \quad (13)$$

where $\varepsilon_n(p_1)$ and $\varepsilon_n'(p_2)$ are mean errors defined by

$$\left. \begin{aligned} \varepsilon_n^2(p_1) &= \sum_{s_1=0}^n \{s_1 - np_1\}^2 F_n(p_1; s_1), \\ \varepsilon_n'^2(p_2) &= \sum_{s_2=0}^n \{s_2 - np_2\}^2 F_n'(p_2; s_2). \end{aligned} \right\} \quad (14)$$

For we have by Tchebycheff's criterion

$$\begin{aligned} \sum_{s_1} F_n(p_1; s_1) &> 1 - \frac{\varepsilon_n^2(p_1)}{n^2 \lambda_1^2}, \\ \sum_{s_2} F_n'(p_2; s_2) &> \frac{\varepsilon_n'(p_2)}{n^2 \lambda_2^2}, \end{aligned}$$

if \sum_{s_1} and \sum_{s_2} denote the summations about s_1 and s_2 respectively satisfying

$$\left| \frac{s_1}{n} - p_1 \right| \leq \lambda_1 \quad \text{and} \quad \left| \frac{s_2}{n} - p_2 \right| > \lambda_2,$$

and therefore

$$\sum_{s_1} F_n(p_1; s_1) - \sum_{s_2} F_n'(p_2; s_2) > 1 - \left\{ \frac{\varepsilon_n^2(p_1)}{n^2 \lambda_1^2} + \frac{\varepsilon_n'^2(p_2)}{n^2 \lambda_2^2} \right\}. \quad (15)$$

On the other hand $\sum_{s_1} F_n(p_1; s_1)$ and $\sum_{s_2} F_n'(p_2; s_2)$ may be written in the form

$$\begin{aligned} \sum_{s_1} F_n(p_1; s_1) &= \sum_{s_1} \sum_{s_2=0}^{n-s_1} F_n(p_1, p_2; s_1, s_2), \\ &= \sum_{s_1} \sum_{s_2} F_n(p_1, p_2; s_1, s_2), \end{aligned} \quad (16)$$

$$\begin{aligned} \sum_{s_2} F_n'(p_2; s_2) &= \sum_{s_2} \sum_{s_1=0}^{n-s_2} F_n(p_1, p_2; s_1, s_2), \\ &= \sum_{s_2} \sum_{s_1} F_n(p_1, p_2; s_1, s_2), \end{aligned} \quad (17)$$

if $F_n(p_1, p_2; s_1, s_2)$ denote the probability of the existence of the numbers s_1 and s_2 of the happening of the events w_1 and w_2 respectively, \sum_{s_1} and \sum_{s_2}

(1) A result which can be looked upon as an extension of Tchebycheff's criterion.

Σ'' denoting the complementary summations of Σ'' and Σ' .

 s_1 s_2 s_1

Since

$$\sum_{s_1} \sum_{s_2} \Sigma'' = \sum_{s_2} \sum_{s_1} \Sigma',$$

we get by (15), (16) and (17)

$$\left(\sum_{s_1} \sum_{s_2} \Sigma' - \sum_{s_1} \sum_{s_2} \Sigma'' \right) F_n(p_1, p_2; s_1, s_2) > 1 - \left\{ \frac{\varepsilon_n^2(p_1)}{n^2 \lambda_1^2} + \frac{\varepsilon_n'^2(p_2)}{n^2 \lambda_2^2} \right\},$$

which gives

$$\sum_{s_1} \sum_{s_2} F_n(p_1, p_2; s_1, s_2) > 1 - \left\{ \frac{\varepsilon_n^2(p_1)}{n^2 \lambda_1^2} + \frac{\varepsilon_n'^2(p_2)}{n^2 \lambda_2^2} \right\}, \quad (18)$$

of which the left-hand side is the probability of the existence of the inequalities

$$\left| \frac{s_1}{n} - p_1 \right| \leq \lambda_1 \quad \text{and} \quad \left| \frac{s_2}{n} - p_2 \right| \leq \lambda_2.$$

Having established the relation (18) it is now easy to show that, if p_1 and p_2 denote the unknown probabilities of w_1 and w_2 and if it be known that the numbers s_1 and s_2 of the happening of w_1 and w_2 respectively satisfy

$$\left| \frac{s_1}{n} - P_1 \right| \leq \lambda_1, \quad \left| \frac{s_2}{n} - P_2 \right| \leq \lambda_2,$$

where P_1, P_2 are positive constants whose sum is less than 1 and λ_1, λ_2 are two given positive numbers subject to the conditions

$$\lambda_s < P_s, \quad (s=1, 2)$$

$$P_1 + P_2 < 1 - (\lambda_1 + \lambda_2),$$

then the probability π_n that the inequalities

$$- \lambda_1 \leq p_1 - P_1 \leq \lambda_1 \quad \text{and} \quad - \lambda_2 \leq p_2 - P_2 \leq \lambda_2$$

hold approaches 1 as limit when n becomes infinite, provided that

$$\left. \begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 \frac{\varepsilon_n^2(p_1)}{n^2} dp_1 &\rightarrow 0 \\ \lim_{n \rightarrow \infty} \int_0^1 \frac{\varepsilon_n'^2(p_2)}{n^2} dp_2 &\rightarrow 0. \end{aligned} \right\} \quad (19)$$

and

The proof can be carried out in a similar manner as in § 1 and therefore we shall not enter into the detailed discussion.

In the general case we obtain, corresponding to (18),

$$\sum_{s_1} \sum_{s_2} \dots \sum_{s_{l-1}} F_n(p_1, p_2, \dots, p_{l-1}; s_1, s_2, \dots, s_{l-1}) > 1 - \sum_{r=1}^{l-1} \left\{ \frac{\varepsilon_n^{(r)}(p_r)}{n \lambda_r} \right\}^2, \quad (20)$$

where the symbols have analogous meanings defined for the case when $l=3$, and we arrive at a sufficient condition for the validity of our theorem that

$$\lim_{n \rightarrow \infty} \int_0^1 \left\{ \frac{\varepsilon_n^{(r)}(p)}{n} \right\}^2 dp \rightarrow 0. \quad (r=1, 2, 3, \dots, l-1) \quad (21)$$

5. Any value of p satisfying $|p-P| \leq \lambda$ may be written in the form

$$p = P + \theta \lambda \quad \text{where} \quad 1 - \leq \theta \leq 1$$

and by putting

$$p_r = P_r + \theta_r \lambda_r, \quad (r=1, 2, \dots, l-1) \quad (22)$$

we have

$$p_l = P_l - \sum_{r=1}^{l-1} \theta_r \lambda_r, \quad (23)$$

if p_l satisfies (10) and P_l be taken to satisfy

$$P_1 + P_2 + \dots + P_l = 1. \quad (24)$$

Accordingly if we attribute a value w_r of an unknown quantity to the event w_r considered in the last article we get, from (22) and (23),

$$\begin{aligned} \left| \sum_{r=1}^l (p_r w_r - P_r w_r) \right| &= \left| \sum_{r=1}^{l-1} \theta_r \lambda_r (w_r - w_l) \right|, \\ &\leq \sum_{r=1}^{l-1} \lambda_r |w_r - w_l|, \end{aligned} \quad (25)$$

so that the probability π_n that a system of inequalities

$$P_s - \lambda_s \leq p_s \leq P_s + \lambda_s$$

holds is not greater than the probability π_n' that the inequality (25) holds. Hence we have the theorem:

In n observations dependent or independent set on the value of an unknown quantity, let s_1 be the number of the happening of the value w_1 , s_2 that of the value w_2 , and so on, s_l that of the value w_l . If it be known that s_1 lies between $n(P_1 - \lambda_1)$ and $n(P_1 + \lambda_1)$, s_2 between $n(P_2 - \lambda_2)$ and $n(P_2 + \lambda_2)$, and so on, s_{l-1} between $n(P_{l-1} - \lambda_{l-1})$ and $n(P_{l-1} + \lambda_{l-1})$, where

P_1, P_2, \dots, P_{l-1} , are positive constants whose sum is less than 1 and $\lambda_1, \lambda_2, \dots, \lambda_{l-1}$ are given positive numbers subject to the conditions

$$\lambda_r < P_r, \quad (r=1, 2, \dots, l-1)$$

$$\sum_{r=1}^{l-1} P_r < 1 - \sum_{r=1}^{l-1} \lambda_r,$$

and if p_1, p_2, \dots, p_l denote the unknown probabilities of the exclusive events w_1, w_2, \dots, w_l such that

$$p_1 + p_2 + \dots + p_l = 1,$$

then the probability π_n' that the inequality

$$\begin{aligned} |p_1 w_1 + p_2 w_2 + \dots + p_l w_l - P_1 w_1 - P_2 w_2 - \dots - P_l w_l| \\ \leq \sum_{r=1}^{l-1} \lambda_r |w_r - w_l|, \end{aligned} \quad (26)$$

holds, where

$$P_l = 1 - (P_1 + P_2 + \dots + P_{l-1}),$$

approaches 1 as limit when n becomes infinite, provided that the condition (21) is satisfied.

Since the theorem holds however small $\lambda_1, \lambda_2, \dots, \lambda_{l-1}$ may be, we have a result which may be expressed in popular language thus:

If n be very large and $\frac{s_1}{n}, \frac{s_2}{n}, \dots, \frac{s_l}{n}$ lie in the neighbourhood of P_1, P_2, \dots, P_l respectively, then it is tolerably probable that $p_1 w_1 + p_2 w_2 + \dots + p_l w_l$ lies in the neighbourhood of $P_1 w_1 + P_2 w_2 + \dots + P_l w_l$, provided that the conditions (21) are satisfied⁽¹⁾.

6. In the last article we have proved, from the knowledge of the relations

$$\left| \frac{s_r}{n} - P_r \right| \leq \lambda_r, \quad (r=1, 2, \dots, l-1) \quad (27)$$

for all of the actual numbers $s_r (r=1, 2, \dots, l-1)$ of the happening of the values w_r , that the probability π_n' of the existence of the inequality

$$\left| \sum_{r=1}^l p_r w_r - \sum_{r=1}^l P_r w_r \right| \leq \lambda, \quad \lambda = \sum_{r=1}^{l-1} \lambda_r |w_r - w_l| \quad (28)$$

for unknown probabilities $p_r (r=1, 2, \dots, l)$ tends to unity when $n \rightarrow \infty$ if certain conditions be satisfied. But we may start, instead of the relation (27), from

(¹) This can be looked upon as a modified extension of Laplace's theorem on independent observations.

$$\left| \sum_{r=1}^l \frac{s_r w_r}{n} - \sum_{r=1}^l P_r w_r \right| \leq \lambda,$$

to arrive at the same results if certain different conditions be satisfied.

In this case we consider the mean error defined by

$$\begin{aligned} \varepsilon_n^2(p_1, p_2, \dots, p_{l-1}) \\ = \sum_s \left(\sum_{r=1}^l s_r w_r - n \sum_{r=1}^l p_r w_r \right)^2 F_n(p_1, p_2, \dots, p_{l-1}; s_1, s_2, \dots, s_{l-1}), \end{aligned} \quad (29)$$

\sum_s denoting the summation about s_1, s_2, \dots, s_l satisfying

$$s_1 + s_2 + \dots + s_l = n. \quad (30)$$

Then by Tchebycheff's criterion

$$\sum_1 F_n(p_1, p_2, \dots, p_{l-1}; s_1, s_2, \dots, s_{l-1}) > 1 - \frac{\varepsilon_n^2(p_1, p_2, \dots, p_{l-1})}{n^2 \lambda^2},$$

where \sum_1 denotes the summation about s_1, s_2, \dots, s_{l-1} satisfying

$$\left| \sum_{r=1}^l \frac{s_r w_r}{n} - \sum_{r=1}^l p_r w_r \right| \leq \lambda,$$

and (30), *i. e.* satisfying

$$\left| \sum_{r=1}^{l-1} \frac{s_r (w_r - w_l)}{n} - \sum_{r=1}^{l-1} p_r (w_r - w_l) \right| \geq \lambda. \quad (31)$$

On the other hand, the probability π_n' is given by

$$\pi_n' = \frac{\alpha_n}{\beta_n},$$

where α_n and β_n are some intermediate numbers between the upper and the lower integrals of

$$\int_{(r)} \sum' F_n(p_1, p_2, \dots, p_{l-1}; s_1, s_2, \dots, s_{l-1}) dp_1 dp_2 \dots dp_{l-1},$$

and

$$\int_{(s)} \sum' F_n(p_1, p_2, \dots, p_{l-1}; s_1, s_2, \dots, s_{l-1}) dp_1 dp_2 \dots dp_{l-1},$$

respectively, if \sum' denotes the summation satisfying

$$\left| \sum_{r=1}^{l-1} \frac{s_r (w_r - w_l)}{n} - \sum_{r=1}^{l-1} P_r (w_r - w_l) \right| \leq \lambda,$$

γ being the domain in $l-1$ dimensions defined by

$$\left| \sum_{r=1}^{l-1} p_r (w_r - w_l) - \sum_{r=1}^{l-1} P_r (w_r - w_l) \right| \leq \lambda,$$

$$p_1 \geq 0, \quad p_2 \geq 0, \quad \dots, \quad p_{l-1} \geq 0, \\ p_1 + p_2 + \dots + p_{l-1} \leq 1,$$

i. e. the domain bounded by two parallel planes

$$\sum_{r=1}^{l-1} p_r (w_r - w_l) - \sum_{r=1}^{l-1} P_r (w_r - w_l) = \pm \lambda, \quad (32)$$

the coordinate-planes $p_1=0, p_2=0, \dots, p_{l-1}=0$ and also the plane

$$p_1 + p_2 + \dots + p_{l-1} = 1, \quad (33)$$

while δ is the domain in $l-1$ dimensions defined by

$$p_1 \geq 0, \quad p_2 \geq 0, \quad \dots, \quad p_{l-1} \geq 0, \\ p_1 + p_2 + \dots + p_{l-1} \leq 1,$$

i. e. the domain bounded by the coordinate-planes and the plane (33).

Now Σ' is replaced by the summation about s_1, s_2, \dots, s_{l-1} satisfying

$$\sum_{r=1}^{l-1} (P_r - p_r) (w_r - w_l) - \lambda \leq \sum_{r=1}^{l-1} \frac{s_r (w_r - w_l)}{n} - \sum_{r=1}^{l-1} p_r (w_r - w_l) \\ \leq \sum_{r=1}^{l-1} (P_r - p_r) (w_r - w_l) + \lambda, \quad (34)$$

of which, for a point $(p_1, p_2, \dots, p_{l-1})$ within γ , the right-hand expression is positive and the left-hand expression is negative, while, for a point $(p_1, p_2, \dots, p_{l-1})$ without γ , the two expressions have the same sign. It follows as in § 1 that

$$\lim_{n \rightarrow \infty} \pi_n' \rightarrow 1,$$

if the condition

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \int_{(\delta)} \epsilon_n^2(p_1, p_2, \dots, p_{l-1}) dp_1 dp_2 \dots dp_{l-1} \rightarrow 0 \quad (35)$$

be satisfied.

I have shown in my previous paper⁽¹⁾ that if $f_p(m, j; m+r, K)$ denotes the probability of the concurrence of w_j in the m^{th} observation and w_k in the $(m+r)^{\text{th}}$ observation, then the mean error defined by (29) will be written in the form

$$\epsilon_n^2(p_1, p_2, \dots, p_{l-1}; s_1, s_2, \dots, s_{l-1}) \\ = n \left\{ \sum_{j=1}^l p_j w_j^2 - \left(\sum_{j=1}^l p_j w_j \right)^2 \right\} + 2V,$$

(¹) L. c. p. 39. There has been treated a more general case, as a special one of which the present case occurs.

where

$$V = \sum w_j w_k \left\{ f_p(m, j; m+r, K) - p_j p_k \right\},$$

the summation being extended over

$$\begin{aligned} j &= 1, 2, \dots, l, & k &= 1, 2, \dots, l, \\ r &= 1, 2, \dots, n-m, & m &= 1, 2, \dots, n-1. \end{aligned}$$

Accordingly the condition (35) is satisfied when the relation

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \int_{(\delta)} \sum w_j w_k |f_p(m, j; m+r, k) - p_j p_k| dp_1 dp_2 \dots dp_{l-1} \rightarrow 0 \quad (36)$$

exists, p_l being to be replaced by

$$1 - (p_1 + p_2 + \dots + p_{l-1}).$$

The new condition (36) also gives rise to several sufficient conditions for the validity of our theorem which are analogous to those found in § 3. Among them I mention here especially the case where any two observations r_1^{th} and r_2^{th} are independent to each other, if $|r_1 - r_2|$ exceeds a definite finite number which is independent of p_1, p_2, \dots, p_{l-1} , as the case will be the simplest one which satisfies the relation (36).

Tôkyô, July 1917.

Ein von Brunn vermuteter Satz über konvexe Flächen und eine Verallgemeinerung der Schwarzschen und der Tchebycheffschen Ungleichungen für bestimmte Integrale,

von

MATSUSABURÔ FUJIWARA in Sendai.

Es sei (K) die Mittelfläche zweier konvexen Flächen (K_1) und (K_2) , und seien O , O_1 und O_2 die Oberflächeninhalte von (K) , (K_1) , (K_2) . Über die Beziehung zwischen O , O_1 und O_2 vermutete Herr Brunn⁽¹⁾, dass die Gültigkeit der Ungleichung $O \geq O_1$, im Falle $O_1 = O_2$, sehr wahrscheinlich sei. Im folgenden werde ich mittelst der Hilbertschen Theorie der linearen Integralgleichungen diese Brunn'sche Vermutung bestätigen.

Inzwischen habe ich bemerkt, dass diese Ungleichung einer Klasse der Ungleichungen von der Form

$$\int f_1 \varphi_1 dx \int f_2 \varphi_2 dx \geq \int f_1 \varphi_2 dx \int f_2 \varphi_1 dx$$

gehört. Die Betrachtung der letzteren Ungleichung hat mich zu einem Satz geführt, der den Schwarzschen und den Tchebycheffschen Satz als speziellen Fall enthält. Diesen Satz und seine einigen Folgerungen möchte ich hier hinzufügen.

1. Nach der Minkowskischen Theorie⁽²⁾ der gemischten Volumen haben wir

$$O_1 = 3V(K_1, K_1, K_0), \quad O_2 = 3V(K_2, K_2, K_0),$$

wobei $V(S_1, S_2, S_3)$ das gemischte Volumen von drei konvexen Flächen (S_1) , (S_2) , (S_3) bedeutet und (K_0) eine Einheitskugel darstellt. Ferner

$$\begin{aligned} O &= 3V(K, K, K_0) = 3V\left(\frac{K_1 + K_2}{2}, \frac{K_1 + K_2}{2}, K_0\right) \\ &= \frac{3}{4} \left\{ V(K_1, K_1, K_0) + 2V(K_1, K_2, K_0) + V(K_2, K_2, K_0) \right\}, \end{aligned}$$

d.h. $4O = O_1 + O_2 + 6V(K_1, K_2, K_0).$

(1) Brunn: Über Ovale und Eiflächen, Münchener Dissertation, 1887, S. 31.

(2) Minkowski: Volumen und Oberfläche, Math. Ann. 57, 1903.

Daher ist die Ungleichung

$$2\sqrt{O} \geq \sqrt{O_1} + \sqrt{O_2}$$

mit

$$3V(K_1, K_2, K_0) \geq \sqrt{O_1 O_2},$$

d. h.

$$V(K_1, K_2, K_0)^2 \geq V(K_1, K_1, K_0)V(K_2, K_2, K_0)$$

äquivalent; oder nach der Hilbertschen Schreibweise mit

$$V(H_1, H_2, R)^2 \geq V(H_1, H_1, R)V(H_2, H_2, R)$$

äquivalent.

Herr Prof. Hilbert⁽¹⁾ ist von der Differentialgleichung

$$L(\mathcal{Q}) + \lambda \frac{(H, H)}{H} \mathcal{Q} = 0, \quad (L(\mathcal{Q}) = (W, H))$$

ausgegangen und die Minkowskische quadratische Ungleichung

$$V(H_1, H_1, H_2)^2 \geq V(H_1, H_1, H_1)V(H_1, H_2, H_2)$$

als Folge seiner Theorie der linearen Integralgleichungen abgeleitet. Da (H, R) die Summe der beiden Hauptkrümmungsradien der durch H charakterisierten Fläche darstellt, und folglich eine positive Funktion ist, kann man die Hilbertsche Theorie auf

$$L_0(\mathcal{Q}) + \lambda \frac{(H, R)}{H} \mathcal{Q} = 0, \quad (L_0(\mathcal{Q}) = (W, R))$$

anwenden. Wenn man daher die Hilbertsche Schlussweise Schritt für Schritt verfolgt, so gelangt man sofort zu dem folgenden allgemeinen

Satz: Sind O, O_1, O_2 die Oberflächeninhalte von drei konvexen Flächen $(K), (K_1)$ und (K_2) , für welche die Relation

$$K = tK_1 + (1-t)K_2, \quad (0 \leq t \leq 1)$$

besteht, so wird stets die Ungleichung

$$\sqrt{O} \geq t\sqrt{O_1} + (1-t)\sqrt{O_2}$$

erfüllt, worin das Gleichheitszeichen gilt nur wenn $(K_1), (K_2)$ homothetisch sind.

2. Versteht man unter H_1, H_2, H_3 die Stützebenenfunktionen von drei vollkommenen Ovaloide $(K_1), (K_2), (K_3)$, so ist nach Minkowski

(¹) Hilbert: Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, 1912, S. 242-257.

$$V(K_1, K_2, K_3) = \frac{1}{6} \int H_1(R_2 T_3 - 2S_2 S_3 + R_3 T_2) d\omega,$$

und das Volumen V_i und der Oberflächeninhalt O_i von (K_i) werden durch

$$V_i = 3V(K_i, K_i, K_i) = \int H_i(R_i T - S_i^2) d\omega,$$

$$O_i = 3V(K_i, K_i, K_0) = \frac{1}{2} \int H_i(R_i + T_i) d\omega$$

dargestellt, wobei (K_0) eine Einheitskugel und $R_i T_i - S_i^2$, $R_i + T_i$ das Produkt und bezw. die Summe der beiden Hauptkrümmungsradien von (K_i) bedeuten. Also sind die Ungleichung

$$V(K_1, K_2, K_0)^2 \geq V(K_1, K_1, K_0) V(K_2, K_2, K_0),$$

und die Minkowskische quadratische Ungleichung

$$V(K_1, K_1, K_2)^2 \geq V(K_1, K_1, K_1) V(K_1, K_2, K_2)$$

in der Form

$$\begin{aligned} & \int H_1(R_1 + T_1) d\omega \int H_2(R_2 + T_2) d\omega \\ & \leq \int H_1(R_2 + T_2) d\omega \int H_2(R_1 + T_1) d\omega, \\ & \int H_1(R_1 T_1 - S_1^2) d\omega \int H_2(R_2 T_2 - S_2^2) d\omega \\ & \leq \int H_1(R_2 T_2 - S_2^2) d\omega \int H_2(R_1 T_1 - S_1^2) d\omega \end{aligned}$$

ausdrückbar.

Mein erster Versuch dies direkt nachzuweisen blieb ohne Erfolg; aber als Nebenprodukt habe ich eine Reihe der neuen interessanten Ungleichungen für bestimmte Integrale gefunden.

Also werde ich mich hier mit der binreichenden Bedingung für eine allgemeine Ungleichung von der Form

$$\int f_1 \varphi_1 dx \int f_2 \varphi_2 dx \geq \int f_1 \varphi_2 dx \int f_2 \varphi_1 dx$$

beschäftigen. Dazu bemerke ich zuerst folgendes.

Die Schwarzsche Ungleichung

$$\int_a^b {}^2(x) dx \int_a^b \varphi^2(x) dx \geq \left(\int_a^b f(x) \varphi(x) dx \right)^2,$$

die in der Analysis bedeutende Rolle spielt, wird bekanntlich aus der Relation

$$\int_a^b \int_a^b \left| \frac{f(x)}{\varphi(x)} \frac{f(y)}{\varphi(y)} \right| dx dy \geq 0$$

abgeleitet. Als ihre Verallgemeinerung ergibt sich die Ungleichung⁽¹⁾

$$\left| \int_a^b f_i(x) f_k(x) dx \right| \geq 0$$

aus der Relation

$$\int_a^b \int_a^b \dots \int_a^b |f_i(x_k)|^2 dx_1 dx_2 \dots dx_n \geq 0.$$

Andererseits hat Herr Franklin⁽²⁾ aus der Betrachtung des Doppelintegrals

$$\int_a^b \int_a^b \left| \frac{f(x)}{1} \frac{f(y)}{1} \right| \cdot \left| \frac{\varphi(x)}{1} \frac{\varphi(y)}{1} \right| dx dy$$

die Tchebycheffsche Ungleichung bewiesen, welche zuerst in der Hermiteschen Cours (2. Aufl.) mit dem Picardschen Beweise versehen, veröffentlicht wurde.

Die gemeinsame Quelle dieser zwei Ungleichungen liegt in der Identität

$$\int_a^b \dots \int_a^b |f_i(x_k)| |\varphi_i(x_k)| dx_1 \dots dx_n = n! \left| \int_a^b f_i(x) \varphi_k(x) dx \right|,$$

welche von Herren Richardson—W. A. Hurwitz aufgestellt wurde. Geht man nun von der speziellen Relation

$$\begin{aligned} (1) \quad & \frac{1}{2} \int_a^b \int_a^b \left| \frac{f_1(x)}{f_2(x)} \frac{f_1(y)}{f_2(y)} \right| \cdot \left| \frac{\varphi_1(x)}{\varphi_2(x)} \frac{\varphi_1(y)}{\varphi_2(y)} \right| dx dy \\ & = \int_a^b f_1 \varphi_1 dx \int_a^b f_2 \varphi_2 dx - \int_a^b f_1 \varphi_2 dx \int_a^b f_2 \varphi_1 dx \end{aligned}$$

aus, und berücksichtigt in (1) die Beziehung

$$\begin{aligned} & \left| \frac{f_1(x)}{f_2(x)} \frac{f_1(y)}{f_2(y)} \right| \cdot \left| \frac{\varphi_1(x)}{\varphi_2(x)} \frac{\varphi_1(y)}{\varphi_2(y)} \right| \\ & = f_2(x) f_2(y) \varphi_2(x) \varphi_2(y) \left(\frac{f_1(x)}{f_2(x)} - \frac{f_1(y)}{f_2(y)} \right) \left(\frac{\varphi_1(x)}{\varphi_2(x)} - \frac{\varphi_1(y)}{\varphi_2(y)} \right), \end{aligned}$$

(¹) Richardson-W. A. Hurwitz: Note on determinants whose terms are certain integrals, Bull. Amer. Math. Society, 16, 1909-10.

(²) Franklin: Proof of a theorem of Tchebycheff's on definite integrals, Amer. Math. Journ. 7, 1885.

so kann man unmittelbar behaupten den

Satz: Ist $f_2(x)\varphi_2(x) > 0$ in (a, b) , und besitzen

$$\frac{f_1(x)}{f_2(x)} - \frac{f_1(y)}{f_2(y)} = F(x, y), \quad \frac{\varphi_1(x)}{\varphi_2(x)} - \frac{\varphi_1(y)}{\varphi_2(y)} = \Phi(x, y)$$

für alle x, y in (a, b) stets dasselbe (oder verschiedenen) Vorzeichen, dann ist

$$(2) \quad \int_a^b f_1 \varphi_1 dx \int_a^b f_2 \varphi_2 dx \geq (\text{oder} \leq) \int_a^b f_1 \varphi_2 dx \int_a^b f_2 \varphi_1 dx.$$

Insbesondere gilt (2) zum Beispiel wenn $f_2(x)\varphi_2(x) > 0$ ist und beide von $f_1(x)/f_2(x)$, $\varphi_1(x)/\varphi_2(x)$ zunehmende oder abnehmende (oder eine davon zunehmende und das andere abnehmende) Funktionen sind.

Die Schwarzsche und die Tchebycheffsche Ungleichungen entsprechen den Fälle, wo $f_1 = \varphi_1$, $f_2 = \varphi_2$ bzw. $f_2 = \varphi_2 = 1$ sind.

3. Nun werden wir diesen Satz spezialisieren.

Setzt man

$$f_1 = f^a, \quad f_2 = \varphi^r, \quad \varphi_1 = f^\beta, \quad \varphi_2 = \varphi^\delta, \\ f(x), \quad \varphi(x) > 0 \text{ in } (a, b); \quad a, \beta, r, \delta \geq 0,$$

dann ist

$$\frac{f_1(x)}{f_2(x)} = \frac{f^a}{\varphi^r}, \quad \frac{\varphi_1(x)}{\varphi_2(x)} = \frac{f^\beta}{\varphi^\delta},$$

daher, wenn $a/\beta = r/\delta = k$ ist, so ergibt sich $f_1/f_2 = (\varphi_1/\varphi_2)^k$, folglich $F(x, y) \cdot \Phi(x, y) \geq 0$ für alle x, y in (a, b) . Also schliesst man sofort

$$\int_a^b f^{a+\beta} dx \int_a^b \varphi^{r+\delta} dx \geq \int_a^b f^a \varphi^\delta dx \int_a^b f^\beta \varphi^r dx.$$

Ersetzt man hier $f^{a+\beta}$, $\varphi^{r+\delta}$ durch f, φ , so hat man

$$\int_a^b f dx \int_a^b \varphi dx \geq \int_a^b f^p \varphi^s dx \int_a^b f^q \varphi^r dx,$$

wo

$$p = \frac{a}{a+\beta}, \quad q = \frac{\beta}{a+\beta}, \quad r = \frac{r}{r+\delta}, \quad s = \frac{\delta}{r+\delta},$$

d. h.

$$r = p, \quad s = q, \quad p + q = 1,$$

was man aus $a/\beta = r/\delta$ sogleich ersieht. Damit ist der folgende Satz bewiesen.

Sind $f(x), \varphi(x) > 0$ in (a, b) , und $p, q \geq 0, p+q=1$, so ist

$$(3) \quad \int_a^b f dx \int_a^b \varphi dx \geq \int_a^b f^p \varphi^q dx \int_a^b f^q \varphi^p dx.$$

Ersetzt man weiter $f^k \varphi^l, f^l \varphi^k$ für f, φ in (3), so wird sie

$$\int_a^b f^k \varphi^l dx \int_a^b f^l \varphi^k dx \geq \int_a^b f^m \varphi^n dx \int_a^b f^n \varphi^m dx,$$

wo

$$m=kp+lq, \quad n=lq+lp, \quad (p+q=1),$$

d. h. $m+n=k+l$ und m, n innerhalb des Intervalles (k, l) liegen.

Also: Wenn $f(x), \varphi(x) > 0$ sind und k, l, m, n beliebige reelle Zahlen bedeuten, von der Art, dass $k+l=m+n$ und (m, n) innerhalb des Intervalles (k, l) liegen, so ist

$$(4) \quad \int_a^b f^k \varphi^l dx \int_a^b f^l \varphi^k dx \geq \int_a^b f^m \varphi^n dx \int_a^b f^n \varphi^m dx.$$

Substituiert man weiter in (3) f^α, f^β für f und φ , wo α, β beliebige reelle Zahlen bedeuten, so bekommt man

$$(5) \quad \int_a^b f^\alpha dx \int_a^b f^\beta dx \geq \int_a^b f^\gamma dx \int_a^b f^\delta dx, \quad (f(x) > 0)$$

wo

$$\gamma = \alpha p + \beta q, \quad \delta = \alpha q + \beta p, \quad p+q=1,$$

d. h. $\alpha + \beta = \gamma + \delta$ und γ, δ innerhalb des Intervalles (α, β) liegen. Insbesondere

$$(6) \quad \int_a^b f^\alpha dx \int_a^b f^{-\alpha} dx \geq \int_a^b f^\beta dx \int_a^b f^{-\beta} dx, \quad (f(x) > 0)$$

wenn $\alpha > \beta$ ist.

In diesen Ungleichungen (5) und (6) ist das Gleichheitszeichen dann und nur dann gültig, wenn $f(x)$ eine Konstante ist, was leicht ersichtlich ist.

Es ist auch ohne Weiteres klar, dass man in (2)-(6) dx durch $F(x)dx$ ersetzen kann, wo $F(x)$ eine beliebige positive Funktion bedeutet; und ferner sind alle bisherigen Resultate auf mehrfache Integrale auftragbar.

Setzt man zum Beispiel in (6)

$$f(x) = 1 - k^2 \sin^2 x, \quad \alpha = 1, \quad \beta = \frac{1}{2}, \quad a = 0, \quad b = \frac{\pi}{2},$$

dann ist

$$\int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 x) dx \int_0^{\frac{\pi}{2}} \frac{dx}{1 - k^2 \sin^2 x} \\ > \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin x} dx \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} > \frac{\pi^2}{4},$$

d. h.

$$\frac{\pi^2}{4} \frac{1 - k^2}{\sqrt{1 - k^2}} > E(k) K(k) > \frac{\pi^2}{4}.$$

Es folgt auch aus

$$\int_0^\infty e^{-t} t^{a-1} dt = \Gamma(a), \quad (a > 0)$$

die Ungleichung

$$\Gamma(a) \Gamma(\beta) > \Gamma(\gamma) \Gamma(\delta)$$

wenn

$$a + \beta = \gamma + \delta, \quad a > \gamma, \delta > \beta > 0.$$

4. Zum Schluss bemerken wir, dass die Bedingung in dem Satz in § 2 zu beschränkt ist, um die Ungleichungen in der Theorie der konvexen Flächen nachzuweisen. Aber aus jenem Satze kann man soweit schliessen, dass für zwei beliebige konvexe Flächen $(K_1), (K_2)$ die Gültigkeit der Relation

$$\left\{ \frac{H_1(\theta, \varphi)}{H_2(\theta, \varphi)} - \frac{H_1(\theta', \varphi')}{H_2(\theta', \varphi')} \right\} \left\{ \frac{L_1(\theta, \varphi)}{L_2(\theta, \varphi)} - \frac{L_1(\theta', \varphi')}{L_2(\theta', \varphi')} \right\} \geq 0$$

für alle Werte von $\theta, \varphi, \theta', \varphi'$, ausgenommen den Fall, wo $(K_1), (K_2)$ homothetisch sind, unzulässig ist, wobei L_i das Produkt oder die Summe der beiden Hauptkrümmungsradien von (K_i) bedeutet. Für zwei konvexe Kurven $(C_1), (C_2)$, deren Stützgeradenfunktionen und Krümmungsradien $p_1(\theta), p_2(\theta)$ bzw. $\rho_1(\theta), \rho_2(\theta)$ sind, gilt

$$\int_0^{2\pi} p_1 \rho_1 d\theta \int_0^{2\pi} p_2 \rho_2 d\theta \leq \int_0^{2\pi} p_1 \rho_2 d\theta \int_0^{2\pi} p_2 \rho_1 d\theta$$

nach dem Brunnischen Satz

$$\sqrt{A} \geq t \sqrt{A_1} + (1-t) \sqrt{A_2},$$

wo A_1, A_2 die Flächeninhalte von $(C_1), (C_2)$ bedeuten, während A den Flächeninhalt einer konvexen Kurve (C) , deren Stützgeradenfunktion p die Relation

$$p(\theta) = tp_1(\theta) + (1-t)p_2(\theta)$$

erfüllt. Berücksichtigt man daher den Satz in § 2, so ist die Gültigkeit der Relation

$$\left(\frac{p_1(\theta)}{p_2(\theta)} - \frac{p_1(\theta')}{p_2(\theta')} \right) \left(\frac{\rho_1(\theta)}{\rho_2(\theta)} - \frac{\rho_1(\theta')}{\rho_2(\theta')} \right) \geq 0 \quad \left(\begin{array}{c} \text{für alle } \theta, \theta' \\ \text{in } (0, 2\pi) \end{array} \right)$$

unzulässig, ausgenommen den Fall, wo $(C_1), (C_2)$ homothetisch sind.

Sendai, Oktober 1917.

On the Involutes of a Curve and Some Applications of the Method of Moving Trihedral,

by

TSURUICHI HAYASHI, Sendai.

In this note, I will give some properties of the involutes of a curve. Most of them are got by using the method of moving trihedral. Some are new, but some are well known. They may be adopted as illustrative examples of the method.

1. Tangents to the involutes of a twisted curve at the corresponding points (lying on the same tangent to the twisted curve) are all parallel to the principal normal to the curve (at the point of contact of the tangent).

2. Principal normals to the involutes at the corresponding points are parallel to each other and all lie on the rectifying plane of the original curve.

3. Binormals to the involutes at the corresponding points are parallel to each other and all lie on the rectifying plane of the original curve.

4. Centres of curvature of the involutes at the corresponding points lie on a straight line passing through the point on the original curve and parallel to the binormals to the involutes.

5. All the involutes have the same polar line at the corresponding points.

6. All the involutes have the same polar developable.

7. The rectifying developable of the original curve is nothing but the polar developable of the involutes.

8. The necessary and sufficient condition for a twisted curve, that its involutes are plane curves, is that the curve is a cylindrical helix.

9. The change of direction of an involute to a cylindrical helix is proportional to that of the original curve, measured from the directions at the corresponding points.

10. Let us treat the ruled surface, that is the totality of straight lines, each of which passes through a point on a curve and has constant direction-cosines a, b, c with respect to the tangent, principal normal, and binormal at the point on the curve. Then the orthogonal trajectories of the generating lines of the surface are got by measuring $k-as$ units of length along the generating lines from the points on the curve, k being a constant and s being the arc-length of the original curve. It is noteworthy that the length is independent of b and c .

When $a=1$, the generating line is nothing but the tangent line of the original curve, and so the surface is the tangent surface of the original curve and the orthogonal trajectories are the involutes of the curve.

When $a=0$, the generating lines come in the normal plane of the original curve. So the orthogonal trajectories of the straight lines on the normal planes of a curve making constant angles with the principal normals are got by measuring off constant lengths along the straight lines from the curve.

11. Taking an arc of a space curve and the corresponding arcs of its involutes, the points which would be the mean-centres of the arcs of the involutes, if the densities at the points of the arcs be supposed to be proportional to the reciprocals of $k-s$, k being the parameter, all lie in a straight line in general, or else are one and the same point.

12. For the involutes of a cylindrical helix, the curvatures at corresponding points of the involutes are proportional to the reciprocals of $k-s^{(1)}$. Hence for such involutes, the Steinerian mean-centres, namely the "Krümmungs-Schwerpunkte" after Steiner⁽²⁾, of the corresponding arcs of the involutes, lie in a straight line in general, or else are one and the same point.

13. For the involutes of an arc of a plane curve, the same is true. Especially for a closed convex plane curve, of which the moving tangent sweeps out an even multiple of π ($=2m\pi$ say) when it revolves along the curve, the Steinerian mean-centres of the corresponding arcs of the involutes of the curve are one and the same point, which lies on the straight line perpendicular to the initial direction of the moving tangent and passing through the Steinerian mean-centre of the original curve,

(1) This is a characteristic property of the cylindrical helix.

(2) Werke II, pp. 97-159 [p. 122], or Crelle's Journal XXI, pp. 33-63 and pp. 101-133.

from which the point is at the distance $l/2m\pi$, l being the perimeter of the original curve.

If the angle swept out by the moving tangent be an odd multiple of π , the Steinerian mean-centre of the involute traces the straight line perpendicular to the initial direction of the moving tangent and passing through the Steinerian mean-centre of the original curve, the distance between the two points being $2k-l$ divided by the multiple of π .

It can be easily proved that the Steinerian mean-centres of a closed plane curve and its evolute (closed) are one and the same point. So the Steinerian mean-centres of a system of parallel convex closed plane curves coincide with the Steinerian mean-centre of their common evolute and therefore are one and the same point.⁽¹⁾

14. If we treat the family of planes passing through tangents and making constant angles with osculating planes, all the characteristics of the family intersect the given curve. If the angle made by the plane with the osculating plane be equal to 45° or 135° , the characteristic lies in the normal plane and bisects the angles between the principal normal and binormal. If the angle be considered as parameter, the characteristics of such families of planes at one point on the curve form the cone

$$\eta^2 - \zeta^2 = \frac{\tau}{\rho} \xi \zeta,$$

referred to the tangent, principal normal and binormal as ξ , η and ζ axes, ρ and τ being the radii of curvature and torsion of the curve at that point respectively. Hence such cones at all the points of a cylindrical helix are congruent.

15. For the family of planes passing through principal normals and making constant angles with normal planes, all the characteristics of the

⁽¹⁾ It is informed by Prof. Kubota that during his study about the locus of the ordinary mean-centres of parallel convex closed plane curves, he has substantially obtained, but overlooked, the latter part of this statement, and that Prof. F. Schilling has proved the first part of this statement for the curves of constant breadth in the *Zeitschrift für Mathematik und Physik*. Bd. 63 (1914), pp. 67-136 [p. 124]. Prof. Kubota also adds that he has still substantially obtained the theorem: The Steinerian mean-centres of a system of parallel convex closed surfaces are one and the same point, the curvature being that of Gauss, i.e. total curvature. For Prof. Kubota's study on the locus of the ordinary mean-centres of parallel curve, see his paper "Über die Schwerpunkte der konvexen geschlossenen Kurven und Flächen," which is to be published in Vol. 14 of this Journal.

family are parallel to rectifying planes and intersect the principal normals. If the original curve is a Bertrand curve, the angle made by the plane with the normal plane can be so chosen that the distance between the characteristic and rectifying plane is constant for all the points on the curve, and then the relative direction of the characteristic with respect to the tangent, principal normal and binormal is also constant. If the angle made by the planes with the normal planes be considered as parameter, the characteristics of such families of planes at one point on the given curve form the quadric

$$\left(\frac{\xi}{\rho} - \frac{\hat{\xi}}{\tau}\right)\eta = \xi.$$

16. For the family of planes passing through binormals and making constant angles with normal planes, the characteristics of the family are not so peculiarly situated. If the angle made by the planes with the normal planes be considered as parameter, the characteristics of such families of planes at one point on the given curve form the quadric

$$\frac{\xi^2 - \eta^2}{\rho} + \frac{\hat{\xi}\xi}{\tau} + \eta = 0.$$

October 1917.

抄 録 短 評

I. 新 刊 書 目

コゝニ掲載セル書籍ノ或ルモノニ就キテハ更ニ詳ニ紹介スルノ機アルベシ。

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N. C. Grotendorst, *Beginnelsen der differentiaal-en integraalrekening*. Vierde verbeterde druk. Breda, De Koninklijke Militaire Academie, 1916.

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R. J. Kortmulder, *De logische grondslagen der wiskunde* (Diss.) Amsterdam, A. H. Kruyt, 1916.

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K. Merz, *Zur Erkenntnistheorie über Raum und Zahl*, aus Historischem der Steiner'schen Fläche. Croire, Librairie Schuler, 1917, 48 p. Fr. 1.00.

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Prompt, *Recherches analytiques sur les carrés magiques*. Paris, Gauthier-Villars, 1917. Fr. 2.20.

E. B. Skinner, *College algebra*. New York, Macmillan 1917. 8+263 p. \$ 1.50.

O. Stolz und J. A. Gmeiner, *Theoretische Arithmetik*. 2te Abteilung: Die Lehren

von den reellen und von den komplexen Zahlen. 2te Auflage. Leipzig, Teubner, 1915. 8+369 p. M. 12.00.

H. Hancock 著, Elliptic integrals (Mathematical monographs edited by Merriman and Woodward, No. 18). New York, John Wiley and Sons, Inc. and London, Chapman and Hall, Ltd., 1917. 104 p. \$1.25.

此 mathematical monographs ハ既ニ我國ニモ大分使用セラレツ、アリト信ズルガ、要用ナル數學上ノ題目ヲ捕ヘテ、餘リ高尙ナラザル程度ニ於テ簡潔ニ叙述セントスルモノナリ。本篇ハ其第十八ヲ爲スモノニシテ楕圓積分ト楕圓函數トヲ論ズ、此等ヲ應用セントスル物理學者、工學者ニトリテハ恰好ノ著述ナリ、純理ヲ主トシテ研究セントスル數學者ニトリテハ、此舊態ノ楕圓積分論ニテハ固ヨリ満足スル能ハズト雖モ、亦左右ニ備ヘテ參考スルノ機多カルベシ、特ニ楕圓積分表ヲ添ヘタルハ、此ノ表ノ得易カラザル時ニ當リテ誠ニ好キ舉ト云フベシ。ツイデニ此 Monographs ノ中ニテ既ニ出版セラレタルモノヲ掲ゲレバ (1) Smith, History of modern mathematics; (2) Halsted, Synthetic projective geometry; (3) Weld, Determinants; (4) McMahon, Hyperbolic functions; (5) Byerly, Harmonic functions; (6) Hyde, Grassmann's space analysis; (7) Woodward, Probability and theory of errors; (8) Macfarlane, Vector analysis and quaternions; (9) Johnson, Differential equations; (10) Merriman, The solution of equations; (11) Fiske, Functions of a complex variable; (12) Carmichael, The theory of relativity; (13) Carmichael, The theory of numbers; (14) Dickson, Algebraic invariants; (15) Henderson, Mortality laws and statistics; (16) Carmichael, Diophantine analysis; (17) Macfarlane, Ten British mathematicians ナリ (T.H.)

山田幸五郎譯, 幾何光學論文集第一(東北帝國大學科學名著集第八冊), 東京, 丸善株式會社, 1918, 342 p. 2.50 圓。

東北帝國大學ノ企畫セル科學上ノ名著ノ翻譯ハ冊ヲ重マルコトハニ達ス。本冊ハ Gauss, Seidel, Finsterwalder, Abbe-Czapski, Zinken-Sommer, Airy 及 Abbe 等ノ論文集ニシテ幾何光學ノ理論ノ頗ル多ク實用ニ供セラル、今日誠ニ絶好ノ紹介ナリト云フベシ、不相變長岡博士ノ嚴密ナル校閲ヲ經タル忠實ノ翻譯ナレバ世ヲ益スルコト大ナルモノアルベシ。本集既發行ノ分ハ (1) Helmholtz, 力(えねるぎ)ノ保存ニ就テ; (2) Kirchhoff, 發散及吸收論; (3) Helmholtz 及 Thomson, 渦動論集; (4) Gauss, ポテンシャル論及地磁氣論; (5) Green, 電氣學及磁氣學ニ於ケル解析數學ノ應用ニ關スル論文; (6) Lagrange, 解析力學抄ナリ。(T.H.)

生駒萬治, 武田登三共著, 新主義ヲ加味セル幾何學教授法及其實際, 上卷, 東京, 教育研究會, 大正七年 (1918). 152 頁, 90 錢。

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II. 雜 誌 內 容

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Rev. J. J. Milne, The geometrical interpretation of homographic equations and their application to loci and envelopes. W. Hope-Jones, The principles of probability and approximations in arithmetic (continued). D. M. Y. Sommerville, Geometry at infinity.

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E. Delssus, Mémoire sur la théorie des liaisons finies unilatérales (fin). A. Denjoy, Mémoire sur la totalisation des nombres dérivés non sommables (suite).

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M. Fréchet, Le théorème de Borel dans la théorie des ensembles abstraits. E. Turrière, Sur la détermination des surfaces par une relation entre des segments de

normales. B. G. Mikhaïlenko, Sur le mouvement d'une bille de billard. E. Maillet, Sur l'équation indéterminée $a^m + b^m = c^m$ en nombres entiers différents de zéro, quand m est fractionnaire et sur une équation analogue plus générale. M. Fouché, Sur la transformation de Lie.

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C. Michel, Mouvements plans dans lesquels la tangente a une vitesse angulaire constante. E. N. Barisien, Sur les paraboles qui passent par les pieds des normales issues d'un point donné à une ellipse. M. F. Egan, Foyers et asymptotes des coniques et quadratiques. A. Myller, Sur les surfaces d'égale pente. F. Gonseth, Sur le centre des moyennes distances d'un groupe de points en ligne droite. A. Favre, Sur les fonctions homogènes.

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E. S. Fedorov, Application des principes de la nouvelle géométrie à la cristallographie.
I. M. Vinogradov, Nouvelle méthode pour obtenir les expressions asymptotiques des
fonctions numériques.

雜 錄 彙 報

歐 米 諸 大 學 ノ 課 程

北 米 合 衆 國

ね ぶ ら す か 大 學 (1917 夏學期)

ぶれんく Brenke, 微積分 (5), 師範課程 (3), カレッヂ代數及ビ平面三角法 (各 5).

い り の い ず 大 學 (1917 夏學期)

み ら ー G. A. Miller, 群論入門, 方程式論及ビ行列式. えむひ A. Emch, 射影幾何學, 作
圖幾何學. わーりん G. E. Wahlin, 高等微積分學, 積分學. ちってんでん E. W. Chittenden,
微積分學. ふれーりー H. D. Frary, 解析幾何學. ぼーるでん R. E. Borden, 平面三角法. リ
チャードソン C. H. Richardson, カレッヂ代數.

う い す こ ん し ん 大 學 (1917 夏學期)

すりひたー C. S. Slichter, 重學 (5), 流體力學 (3), 代數學 (5). どうりんぐ L. W. Dowl-
ing, 函數論 (3), 高等幾何學 (5), 解析幾何學 (5). どれすでん A. Dresden, 變分學 (3), 定積
分 (5), 微積分 (5). しむぷそん T. M. Simpson, 微分方程式 (5), 商業代數學 (5), 三角法 (5).
はーと W. W. Hart, 行列式 (3), 立體幾何學 (5), 師範課程 (5).

て い ぶ り っ つ ノ 定 理 ニ 就 テ

波蘭數物雜誌 Prace matematyczno-fizyczne, vol. 22, (1911) に於テ Toeplitz が發表シタ
ル數列ニ關シテノ一定理ヲ, 最近ノ本誌 12 卷 4 號ニ於テ, 小島君ガ或方面ニ擴張シ, 且ツ其
應用ニヨリテ多クノ有益ナル結果ヲ得ラレタリ. 此兩氏ノ基本定理ノ間ニハ勿論相類似シタ
ル思想ノ伏在ス可キヲ以テ, 之ヲ適當ノ順序ニ講ジテ成ル可ク證明ノ勞ヲ尠ナカラシメンニハ
如何ニス可キカラ思ヒ見タル際, 二三心ニ浮ブ所ノモノアリ. 下ニ録ス.

先ヅ吾人ハ Toeplitz ト全ク同一ノ思想ヲ以テ, 次ノ定理ヲ證明スル事ヲ得. 只 y_n ノ極
限ヲ 0 ナリトスル事ニヨリテ幾分か證明中ノ式ヲ簡略ニシ得タル所アラン.

(1) 數列 $\{x_n\}$ が收斂スル時, 之ト對應スル數列

$$\{y_n\} = \{a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n\}$$

ガ必ズ 0 ニ收斂スルト云フ事實ガ成立スル爲メニハ, 次ノ三ツノ條件ガ必要ニシテ且ツ十分
ナリ. 即

(a)
$$\lim_{n \rightarrow \infty} (a_{n1} + a_{n2} + \dots + a_{nn}) = 0,$$

$$(b) \quad \lim_{n=\infty} a_{nk}=0, \quad k=1, 2, 3, \dots,$$

$$(c) \quad |a_{n1}| + |a_{n2}| + \dots + |a_{nn}| < M$$

ナル様 = n = 無關係ノ正數 M ラ定メ得ル事。

今此定理ヲ少シク變ジテ $\{x_n\}$ ガ收斂スル時 $\{y_n\}$ モ亦前者ト同一ノ極限ニ向ツテ收斂スルト云フ爲メノ條件ヲ求メンハ、

$$\{y_n - x_n\} = \{a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn}-1)x_n\}$$

ナル數列ガ 0 = 收斂スル爲メノ條件ヲ求ムレバ足レリ。即前定理ヨリ直チニ次ノ三條件ガ必要且ツ十分ナルヲ知ル可シ。

$$(a_1) \quad \lim_{n=\infty} (a_{n1} + a_{n2} + \dots + a_{nn}) = 1,$$

$$(b_1) \quad \lim_{n=\infty} a_{nk} = 0, \quad k=1, 2, 3, \dots,$$

$$(c_1) \quad |a_{n1}| + |a_{n2}| + \dots + |a_{nn}| < M.$$

之即 Toeplitz ノ得タル結果ナリ。若シ此問題ニ於テ定號ノ無限大ニ發散スル場合ヲモ等シク收斂ト云フ言葉ノ内ニ包含セシメ、其定號無限大ヲ以テ數列ノ値ト呼ブナラバ、吾人ノ條件 (c_1) ヲ次ノ如ク變ゼザル可カラズ

(c'_1) n = 關係ナキ一定ノ正整數 m アリテ m 以上ノ任意ノ n = 對シテ

$$a_n, m, a_{n+m+1}, \dots, a_{nn}$$

ハ總テ負數ニアラズ。

更ニ又定理 (1) ノ假定ヲ變ジテ $\{x_n\}$ ガ收斂スル時 $\{y_n\}$ モ亦收斂ス但シ兩者ノ極限ハ必ズシモ等シキヲ要セズト云フ場合ノ條件如何。此時ハ n ト共ニ變化スル任意ノ正整數 $m=f(n)$ = 對シテ $\{y_n - y_{n+m}\}$ ハ 0 = 收斂スル事トナル、逆モ亦成立ス。即任意ノ m ヲ取ル時 $\{y_n - y_{n+m}\}$ ガ定理 (1) ノ條件ヲ備フルト云フ事ガ必要且ツ十分ナリ。此條件ヨリ m ヲ逐出セバ容易ニ次ノ條件ヲ得可シ。即

$$(a_2) \quad \{a_{n1} + a_{n2} + \dots + a_{nn}\} \text{ ガ收斂ス,}$$

$$(b_2) \quad \{a_{nk}\}, k=1, 2, 3, \dots \text{ ガ收斂ス,}$$

$$(c_2) \quad |a_{n1}| + |a_{n2}| + \dots + |a_{nn}| < M.$$

之即小島君ノ得ラレタル基本ノ定理ナリ。

定理 (1) ノ假定ヲ更ニ再ビ變化シテ $\{x_n\}$ ガ收斂スル時 $\{y_n\}$ ハ收斂スルカ又ハ高々有限ノ範圍ヲ振動スルト云フ場合ノ條件ヲ求メンハ、 0 = 收斂スル任意ノ數列 $\{z_n\}$ = 對シテ $\{z_n y_n\}$ ガ定理 (1) ノ條件ヲ備フル事ヲ言ヘバ足レリ。之ヨリ $\{z_n\}$ ヲ逐出セバ求ムル條件ハ次ノ只一ツニテ包括シ得ルヲ知ル

$$(c_3) \quad |a_{n1}| + |a_{n2}| + \dots + |a_{nn}| < M.$$

(c) , (c_1) , (c_2) , (c_3) ハ皆同一ナルコトヲ注意ス可シ。

次ニハ $\{x_n\}$ ナ單ナル數列トセズ、之ニ一ツノ制限ヲ加ヘテ同様ノ研究ヲ施シ得ル一例ヲ掲ゲン。

(2) 單調ニ減小スル正項ノ數列 $\{x_n\}$ (隨テ勿論收斂ス)ニ對シテ

$$\{y_n\} = \{a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n\}$$

ガ必ズ 0 = 收斂スルト云フ事實ガ成立スル爲メニハ、次ノ三ツノ條件ガ必要ニシテ且ツ十分ナリ。即

$$(\alpha) \quad \lim_{n=\infty} (a_{n1} + a_{n2} + \dots + a_{nn}) = 0,$$

$$(\beta) \quad \lim_{n=\infty} a_{nk} = 0, \quad k=1, 2, 3, \dots,$$

$$(\gamma) \quad |a_{n1} + a_{n2} + \dots + a_{nn}| < M$$

ナル様ニ n, m = 無關係ノ正數 M ヲ定メ得ルコト。

此條件ト定理 (1) ノ條件トノ異ナル所ハ只其第三ノ條件ニアルノミ。

定理 (1) ヲ基礎トシテ他ノ變形サレタル條件ヲ得タルト全ク同一ノ步調ヲ以テ、定理 (2) ヨリ發シテ變形サレタル條件ヲ得可シ。即同ジク單調數列 $\{x_n\}$ = 對シ $\{y_n\}$ が同一ノ極限ヲ有スル爲メノ條件トシテハ $(a_1), (b_1), (\gamma)$ ヲ取レバ可ナリ。 $\{y_n\}$ が單ニ收斂スルト云フ丈ノ假定ナラバ條件トシテ $(a_2), (b_2), (\gamma)$ ヲ取レバ可ナリ。 $\{y_n\}$ が高々有限範圍ヲ振動スルト云フ場合ニ對シテハ (γ) 只一ツヲ取レバ足レリ。

此外例ヘバ $\{a_n\}$ ヲ有限の振動數列トセル時ニモ稍々類似セル結果ヲ得。(S. K.)

ひるべるとノ完備ノ公理ニ就イテ

有名ナひるべると教授ノ幾何學原理 (Hilbert, Grundlagen der Geometrie 林教授及ビ小野氏ノ邦譯アリ) = 掲ゲラレタ初等幾何學公理體系ノ第五群ハ誰モ知ル如ク連續ノ公理ト稱セラレレモノデアルガ、此公理群ハ二ツニ分レ、第一ハ所謂あるひめですノ公理デアツテ測定ニ關シ、第二ハ完備ノ公理ト名ケラレタモノ是レデアル。此公理ハ次ノ命題カラ成ル：幾何學ノ要素(點、直線、平面)ハ前述ノ總テノ公理ヲ保持スル以上ハ擴張ヲ許サマル所ノ物ノ一系統ヲ形作ルト。

此命題ノ意味ハ普通あるひめですノ公理ニ至ルマデノ第一カラ第五ニ至ル公理群ニ由ツテ定メラレル要素即チ點、直線、平面ノ系統ガ最早擴張ヲ容サルモノナルコト、即チ此等ノ公理群ヲ満足スル要素系統ノ一ツ以外ニ存在シ得ザルコトデアルト考ヘラレテ居ルヤウデアル。若シ然ラバ此處ニ謂フ完備トハええぶれん (Veblen) ノ斷定的 (categorical) ト名ケタモノト同一デアツテ、現ニ氏ハ斯ク解シテひるべるとノ完備ヲはてゐんとん (Huntington) ノ所謂充足 (sufficient) トイフ語ニ當ルモノト見テ居ルヤウデアル (Veblen, A System of Axioms for Geometry, Trans. Amer. Math. Soc. V. p. 346, 1904)。併シナガラ斷ツテ考ヘルト已ニゞえぶれんノ研究ガ示ス如ク公理系統ガ適當ニ定セラレラバ其斷定的ナルコトハ實際ニ證明セラレルノデアツテ、斷定性ハ公理系統ノ完全ナルタメノ要件トイハナケレバナラナイ。之ヲ公理トシテ掲ゲルノハ纏デナイヤウニ思ハレル。其上若シひるべるとノ意ガ單ニ公理系統ノ斷定性ニアルトスルナラバ、完備ノ公理ヲ連續ノ公理ニ含マシメル理由ハ無イ。余ハひるべるとノ完備ノ公理ヲ唯其命題ノ語句ノ上カラ公理系統ノ斷定性ヲ意味スルト解スルノハ恐ラク氏ノ意ヲ得タルモノデハアルイと思フ。

然ラバ完備ノ公理ガ眞ニ連續ノ公理ノ一部ヲ成ス所以ハ如何ナル點ニアルカ。余ハ此公理ガ直線上ノ有理點ノミナラズ、極限點タル無理點ヲモ幾何學ノ對象トシテ採ルコトヲ述ベルモノデハナイカと思フ。あるひめです公理ニ至ルマデノ第一カラ第五ニ至ル公理系統ハ有理點ノ系列體系デモ満足セラレ、又實數點ノ系列體系デモ満足セラレルノデアツテ實ハ未ダ斷定的トイフコトハ出來ナイ。完備ノ公理ニ由ツテ點系列ノ體系ガ其上ニ擴張ヲ容サルモノナルコトヲ規定スルニ至ツテ、該體系ハ有理點ノ系列體系ナル能ハズシテ實數點ノ系列體系ナラザルベカラザルコトガ確定セラレルノデアル。此公理ハ有理點ナラズ、其系列ノ極限トシテノ無理點ヲモ幾何學ノ對象トシテ採ルコトヲ陳述スルノデアル。斯ク解スルコトニ由ツテ初メテ完備ノ公理ガ連續ノ公理ノ一部トシテあるひめですノ公理ト獨立ニ掲ゲラルベキ所以ガ理解セラレデアラウ。余ハひるべるとノ眞意ガ此處ニアルノデアツテ、單ニ所謂斷定的ノ充足トイフ如キ意味ニ之ヲ解スルノハ恐ラク正鵠ヲ失スルモノデハナイカと思フ。固ヨリ此ノ如キコトハ識者ノ夙ニ知ラレル所ト信ズルケレドモ、余自身が完備ノ公理ノ意味ヲ解スルニ苦シミ、現ニ上ノ如キ解釋ガ廣ク行ハレルノデハナイカト疑フニ由リ、此處ニ一言シテ大方ノ教ヲ乞ヒタイと思フノデアル。(田邊 元)

平均曲線ニ關スル極大極小問題

面積 A_1, A_2 , ヲ有スル二ツノ閉凸曲線 C_1, C_2 ノ平均曲線 C ノ面積ヲ A トスレバ、Brunn ノ定理ニヨリテ $2\sqrt{A} \geq \sqrt{A_1} + \sqrt{A_2}$ ガ成立ス。今 A_1, A_2 ノ値ヲ一定シオキテ C_1 ,

C_2 の形ヲ變化セシムル時、 $A \propto 2\sqrt{A} = \sqrt{A_1} + \sqrt{A_2}$ トナル場合ニ極小トナル、コレハ C_1 、 C_2 ガ相似ニシテ相似ノ位置ニアル時ニ限ラル、コトハ已ニ知ラレテアル。 A ノ極大如何トイフ問題ニ對シテハ、吾々ハ A ニハ極大ナシト答ヘルコトガ出來ル。コレハ C ノ幅ハ同方向ノ C_1 、 C_2 ノ幅ノ和半ニ等シトイフ事實カラ次ノ如クニ證セラレル。

今 C_1 ヲ一定シオキ、 C_2 ヲシテ面積ハ一定ノ値 A_2 ヲ有シツ、次第ニ細長キ棒狀ヲトラシメル。即チ C_2 ノ或ル方向ノ幅 l_1 ヲ大ニシ、之レニ直角ノ方向ノ幅 l_2 ヲ小ナラシムレバ、 l_1 ヲイカニ大ニシテモ l_2 ヲ相應ニ小ニスレバ C_2 ノ面積ヲシテ依然 A_2 ニ止マラシムルコトガ出來ルデアラウ。然ルニ平均曲線 C ノ幅ヲ見ルニ、或ル方向ノ幅ハ l_1 ノ半ヨリ大ニ、之ニ直角ナル方向ノ幅ハ C_1 ノ相當スル幅ノ半ヨリ大デアル。 C_1 ハ一定デアルカラ l_1 サヘ大ナラシムレバ A ナシテイカホドニテモ大ナラシムルコトガ出來ルデアラウ。

ソレデ問題ヲ次ノ如ク少シク變更スル： C_1 ト C_2 トヲ相等シキ曲線トシテ、相互ノ位置ヲ變化セシメタル場合、平均曲線 C ノ面積ノ極大如何。

例ニヨリ C_1 、 C_2 、 C ヲ $p=p_1(\theta)$ 、 $p_2(\theta)$ 、 $p(\theta)$ デ表ハスト、吾々ノ場合ニテハ C_2 ハ C_1 ヲアル角丈ケ廻轉セシメタモノト考ヘラレルカラ、 $p_2(\theta)=p_1(\theta+\alpha)$ トオクコトガ出來ル。

2 $p(\theta)=p_1(\theta)+p_2(\theta)$ ヲリシテ

$$4A = 2 \int_0^{2\pi} p'(\theta) \rho(\theta) d\theta, \quad \rho(\theta) = p(\theta) + p''(\theta),$$

從テ

$$4A = 2A_1 + \int_0^{2\pi} p_1'(\theta+\alpha) \rho_1(\theta) d\theta$$

ガ導カレル。故ニ A ノ極大極小問題ハ、 α ノ函數

$$F(\alpha) = \int_0^{2\pi} p_1'(\theta+\alpha) \rho_1(\theta) d\theta$$

ノソレデアツテ普通ノ極大極小問題ノ形ニ直サレル。故ニ求ムル α ハ

$$\int_0^{2\pi} p_1'(\theta+\alpha) \rho_1(\theta) d\theta = 0$$

ヲ満足シナケレバナラス。然ルニ $p_1(\theta)$ ノ週期ハ一般ニ 2π デアルカラ、

$$\int_0^{2\pi} p_1'(\theta+\alpha) \rho_1(\theta) d\theta = - \int_0^{2\pi} p_1'(\theta) \rho_1(\theta+\alpha) d\theta = - \int_0^{2\pi} p_1'(\theta+\alpha) \rho_1(\theta+2\alpha) d\theta$$

ガ得ラレル。之ヨリ直チニ $\alpha=\pi$ ガ一ツノ解答ヲ與ヘルコト、並ニ $p_1(\theta)$ ガ $\alpha_0 < 2\pi$ ナル週期ヲ有スレバ、 $\alpha=\frac{\alpha_0}{2}$ ガ又一ツノ解答ヲ與ヘルコトガ分ル。併シソレ等ガ極大極小何レニナルカハ一概ニイヘナイ。

定曲線ニアリテハ $\alpha=\pi$ ハ一ツノ極大ニ相當スルコト、正 n 邊形ニアリテハ $\alpha=\frac{\pi}{n}$ ガ極大(絶對的)ニ相當スルコトガ證明セラレルガ、一般ノ場合ニ對シヨリ決定的ノ解答ガ得ラレマイカ。之ヲ一ツノ問題トシテ提出スル。(M.F.)

初等幾何學ノ一問題

本誌前號雜錄ニ於テ余ハ初等幾何學ノ一作圖題ヲ提出セリ。次ノ作圖題モ亦同様ニ三次元ノ空間ノ助ニヨリテ容易ニ解キ得ラルベシト雖、平面上ノ簡單ナル作圖法ハ未ダ之ヲ見出ス能ハザルナリ。

a) 一平面上ニ $A, A_2; B_1, B_2$ ナル二双ノ點ガ與ヘラル、トキ、一双ノ點 X_1, X_2 ヲバ、 $A_1, A_2; X_1, X_2$ ガ一圓周上ニアル二双ノ調和共軛點、 $B_1, B_2; X_1, X_2$ ガ一圓周上ニアル二双

ノ調和共軛點ナル様ニ決定セヨ。

b) 一平面上ニ $A_1, A_2; B_1, B_2; C_1, C_2$ ナル二双ノ點ガ與ヘラル、トキ、一雙ノ點 X_1, X_2 ヲバ、 $A_1, A_2; X_1, X_2$ ガ一圓周上ニアル二雙ノ調和共軛點、 $B_1, B_2; X_1, X_2$ ガ一圓周上ニアリ C_1, C_2, X_1, X_2 ガ一圓周上ニアル様ニ決定セヨ。

射影幾何學的ノ作圖ヲ許ストキハ、前號ノ問題ハ次ノ如クニ解決スルコトヲ得。(a) (b) モ亦同様ニ解キ得ラルベシ。

補題 1. 一平面上ニ三雙ノ點 $A_1, A_2; B_1, B_2; C_1, C_2$ ガ與ヘラル、トキ、一雙ノ動點 X_1, X_2 ガ、三組ノ四點 $A_1, A_2, X_1, X_2; B_1, B_2, X_1, X_2; C_1, C_2, X_1, X_2$ ガ各一圓周上ニアル様ニ動クトキハ、 X_1, X_2 ヲ結び付クル直線ハ三直線 A_1A_2, B_1B_2, C_1C_2 ニ切スル一圓錐曲線 \mathcal{R} ヲ包ムベシ。

前號ニ於テ述ベタル如ク球ノ中心ヨリ平面ヘ下セル垂線ト球トノ交點 O ヨリ圓形ヲ stereographically = 球面上ニ射影シ、 $A_1, A_2; B_1, B_2; C_1, C_2$ ノ射影ヲ夫々 $a_1, a_2; b_1, b_2; c_1, c_2$ トシ、 X_1, X_2 ノ射影ヲ夫レ々 x_1, x_2 トスルトキハ、直線 x_1x_2 ハ三直線 a_1a_2, b_1b_2, c_1c_2 ニ交ハル。故ニ直線 x_1x_2 ハ一ツノ二次線織面上ニアルベク、從ヒテソノ射影ナル直線 X_1X_2 ハ A_1A_2, B_1B_2, C_1C_2 ナル三直線ニ切スル一圓錐曲線 \mathcal{R} ニ切スベシ。

此圓錐曲線 \mathcal{R} ヲ決定センニハ X_1, X_2 ノ二雙ノ異ナレル位置ヲ見出スヲ以テ足レリトス。

A_1, A_2 ヲ過ギル一圓 \mathcal{C} ヲ作り、 B_1, B_2 ヲ過ギル任意ノ二圓ト \mathcal{C} トノ根心ヲ O_1, C_1, C_2 ヲ過ギル任意ノ二圓ト \mathcal{C} トノ根心ヲ O_2 トシ、 O_1O_2 ヲ結び付クル直線ヲ引キ、圓 \mathcal{C} ト二點 X_1, X_2 ニ於テ交ハラシムルトキハ、 X_1, X_2 ハ所要ノ位置ナリ。圓 \mathcal{C} ヲ動カスコトニヨリテ何程ニテモ X_1, X_2 ノ位置ヲ見出スヲ得ベク從ヒテ圓錐曲線 \mathcal{R} ハ決定セラルベシ。

サテ此補題 1 ヲ用キテ前號ノ問題ヲ解カシムルニハ、 $A_1, A_2; B_1, B_2; C_1, C_2$ ニヨリテ圓錐曲線 \mathcal{R} ヲ決定シ、同様ニ $B_1, B_2; C_1, C_2; D_1, D_2$ ニヨリテ圓錐曲線 \mathcal{Q} ヲ決定スルトキハ \mathcal{R} ト \mathcal{Q} トハ既ニ二ツノ共通切線 B_1B_2, C_1C_2 ヲ有ス。依リテ所要ノ點二雙ハ \mathcal{R} ト \mathcal{Q} トノ他ノ共通二切線上ニアリ。カクシテ所要ノ點ハ決定セラルベシ。

補題 2. 一平面上ニ二雙ノ點 $A_1, A_2; B_1, B_2$ ガ與ヘラル、トキ、一雙ノ動點 X_1, X_2 ガ、 $A_1, A_2; X_1, X_2$ ガ一圓周上ニアル調和共軛點、 $B_1, B_2; X_1, X_2$ ガ一圓周上ニアル様ニ動クトキハ X_1, X_2 ヲ結び付クル直線ハ A_1A_2, B_1B_2 ニ切スル一圓錐曲線 \mathcal{R} ヲ包ム。

圓形ヲ點 O ヨリ球面上ニ stereographically = 射影シ、 $a_1, a_2, b_1, b_2, x_1, x_2$ ヲ夫レ々 $A_1, A_2, B_1, B_2, X_1, X_2$ ノ射影トセヨ。然ルトキハ直線 x_1x_2 ハ直線 a_1a_2 及ビソノ線ノ球ニ關スル reciprocal polar 及ビ直線 b_1b_2 ニ交ハル。之レニ依リテソノ射影ナル直線 X_1X_2 ハ A_1A_2, B_1B_2 ナル二直線ニ切スル一圓錐曲線 \mathcal{R} ヲ包ム。

此圓錐曲線 \mathcal{R} ヲ決定センニハ直線 X_1X_2 ノ三ツノ位置ヲ見出スヲ以テ足レリトス。即 A_1, A_2 ヲ過ギル任意ノ圓 \mathcal{C} ヲ作り B_1, B_2 ヲ過ギル任意ノ二圓ト \mathcal{C} トノ根心ヲ O_1 トシ、點 O_1 ヲ圓 \mathcal{C} ニ關スル直線 A_1A_2 ノ極點ニ結び付クル直線ヲ引キ、ソノ直線ガ圓 \mathcal{C} ニ交ハル二點ヲ X_1, X_2 トスレバ是レ所要ノ二點ナリ。圓 \mathcal{C} ヲ動カスコトニヨリテ X_1, X_2 ノ位置ハ何程ニテモ求メラルベシ。カクシテ圓錐曲線 \mathcal{R} ハ決定セラル。

之ヲ用キテ (a) ヲ解カシムルニハ、一雙ノ動點 X_1, X_2 ガ、 $A_1, A_2; X_1, X_2$ ガ一圓周上ニアリ、 $B_1, B_2; X_1, X_2$ ガ一圓周上ニアル調和共軛點ナル様ニ動クトキ、直線 X_1X_2 ハ二直線 A_1A_2, B_1B_2 ニ切スル一圓錐曲線 \mathcal{Q} ヲ包ム。而シテ \mathcal{R} ト \mathcal{Q} トハ既ニ二切線 A_1A_2, B_1B_2 ヲ共有ス。故ニ他ノ二切線ヲ決定スルトキハ所要ノ二點 X_1, X_2 ハ此二切線上ノ二雙ノ點トシテ容易ニ決定セラルベシ、一雙丈ケ實點ナリ。

補題 3. 點 O ヨリ二點 A_1, A_2 ヲ球面上ニ stereographically = 射影シ、二點 a_1, a_2 ヲ得、又直線 a_1a_2 ノ球ニ關スル reciprocal polar ヲ作り、之レヲ點 O ヨリ再平面上ニ射影スルトキハ、ソノ射影ハ線分 A_1A_2 ノ垂直二等分線ナルベシ。

之レヲ證明スルニ當リ球ガ直角坐標ノ原點ヲ中心トシ單位半徑ヲ有スルモノトシ xy 平面ヲ考フル所ノ平面ナリトシテ論ズルモ問題ヲ制限スルコトナカルベシ。

xy 平面上ノ點 $A_1(\xi, \eta), A_2(\xi', \eta')$ ノ射影 a_1, a_2 ノ坐標ハ

$$\frac{2\xi}{\xi^2+\eta^2+1}, \frac{2\eta}{\xi^2+\eta^2+1}, \frac{\xi^2+\eta^2-1}{\xi^2+\eta^2+1}; \frac{2\xi'}{\xi'^2+\eta'^2+1}, \frac{2\eta'}{\xi'^2+\eta'^2+1}, \frac{\xi'^2+\eta'^2-1}{\xi'^2+\eta'^2+1}$$

ナルヲ以テ、球=關スル直線 a_1, a_2 ノ reciprocal polar ハニツノ方程式

$$\frac{2\xi}{\xi^2+\eta^2+1}x + \frac{2\eta}{\xi^2+\eta^2+1}y + \frac{\xi^2+\eta^2-1}{\xi^2+\eta^2+1}z = 1. \quad (1)$$

$$\frac{2\xi}{\xi'^2+\eta'^2+1}x + \frac{2\eta'}{\xi'^2+\eta'^2+1}y + \frac{\xi'^2+\eta'^2-1}{\xi'^2+\eta'^2+1}z = 1 \quad (2)$$

ニ依リテ與ヘラル。今之レヲ $(0, 0, 1)$ ナル點 O ヨリ xy 平面上ニ射影シテ得タル直線ノ方程式ヲ求メンニ、(1), (2) 及

$$\frac{x}{x'} = \frac{y}{y'} = 1 - z$$

ヨリ x 及 y ナ消去セバ得ラル。

$$\begin{aligned} \text{即} \quad & \frac{2\xi}{\xi^2+\eta^2+1}x'(1-z) + \frac{2\eta}{\xi^2+\eta^2+1}y'(1-z) + \frac{\xi^2+\eta^2-1}{\xi^2+\eta^2+1}z = 1, \\ & \frac{2\xi'}{\xi'^2+\eta'^2+1}x'(1-z) + \frac{2\eta'}{\xi'^2+\eta'^2+1}y'(1-z) + \frac{\xi'^2+\eta'^2-1}{\xi'^2+\eta'^2+1}z = 1 \end{aligned}$$

ヨリ z ヲ消去セバ所要ノ結果トシテ

$$(\xi - \xi')x + (\eta - \eta')y - \frac{\xi^2 + \eta^2 - \xi'^2 - \eta'^2}{2} = 0$$

ヲ得。之レ A_1, A_2 ナル線分ノ垂直二等分線ナリ。

系。之レニヨリテ補題 2 ノ圓錐曲線 \mathcal{R} ハ線分 A_1, A_2 ノ垂直二等分線ニ切ス。

サテ (b) ヲ解カンニ X_1, X_2 ナル動點ガ、 A_1, A_2 ; X_1, X_2 ハ一圓周上ニアルニ双ノ調和共軛點、 C_1, C_2, X_1, X_2 ガ一圓周上ニアル様ニ動クトキハ、二點 X_1, X_2 ヲ結ビ付クル直線ハ直線 A_1, A_2 及ビツノ垂直二等分線ニ切スル一圓錐曲線 \mathcal{Q} ヲ包ム。是ニ依リテ \mathcal{R}, \mathcal{Q} ナル二圓錐曲線ハ、ニツノ共通切線 A_1, A_2 及ビ線分 A_1, A_2 ノ垂直二等分線ヲ有ス。故ニ所要ノ二双ノ二點ハ、此等ノ二圓錐曲線ノ他ノ二共通切線上ノ點トシテ容易ニ決定セラルベシ。(T.K.)

素 數 = 關 ス ル 一 定 理

10 ノ正整數羈ガニツノ素數 a, b ノ和ニ變形セラレ、其何レカガ 3 ナラザルトキハ $a \equiv b \equiv 1 \pmod{3}$ ナリト、大阪ノ山田光雄氏報ゼラル。

東 京 數 學 物 理 學 會 特 別 講 演 會

同會第十八回ハ東京帝國大學理科學部中央講堂ニ於テ大正 6 年 (1917) 11 月 15 日 (木曜日) 午後 6 時半ヨリ約二時間半開カル。其演題、講師並ニ講演要項次ノ如シ

量子論 (quanta-theory)

理學博士 石 原 純氏

1. 量子論ノ起源
2. 量子論ノ應用概觀
3. 量子假說ノ意義ノ探究
 - a. 統計力學ニ於ケル量子假說
 - b. 力學ニ於ケル熱力學の類推
 - c. 恒熱の不變量 (adiabatic invariant) トシテノ作用量子
 - d. 制限の週期運動 (conditionally periodic motion) ノ特性

4. 結論

同會第十九回ハ同處ニ於テ大正 6 年 (1917) 12 月 20 日 (木曜日) 午後 6 時半ヨリ約 2. 時間開カル。次ノ如シ

放射性物質ノ環境系統ト終局物質

理學博士 木下 季吉氏

1. 放射性物質
2. 原子環境説
3. 環境系統
4. α 線ノ透過距離ト環境係數トノ關係
5. 生成物ノ原子量
6. 放射性物質ノ週期律表中ニ於ケル位置ト環境定型トノ關係
7. Atomic number.
8. Isotopism.
9. 終局物質

同會第 20 回ハ大正 7 年 (1918) 2 月 21 日理學博士寺田寅彦氏ノ原子構造説概觀ノ講演ニテ開カル、由ナリ。

再 ビ 日 本 數 學 史 ニ 就 テ

Smith 及ビ 三上兩氏共著トシテ頗ル趣味アル History of Japanese Mathematics ニツキテハ、本誌第 11 卷第 187 頁ニ掲載セシガ如ク、特ニ本誌監修者林鶴一ニ於テ其ノ割引販賣ノ取扱ヲナスベシ。普通價格ハ \$3 卽六圓餘ナルガ割引價格ハ四圓五十錢ナリ。迅速申込マルベシ。

本 誌 ノ 既 刊 號 ニ 就 テ

東北數學雜誌第一卷第一及第二號ハ既ニ餘程以前ニ賣切レトナリ居ルガ、特志家ノ要求頗ル多キニ因テ、研究援助ノ目的ニテ、所藏不要ノ御方ハ此等要求者ニ融通提供セラレタシ。即チ本誌監修者林鶴一ニ於テ其ノ中介ヲナス爲メニ相當代價ヲ以テ購入スベシ。第一卷第三號以下第五卷第四號ニ至ル間ノ各號モ亦同様ノ取扱ヲナスベシ。

二 三 雜 誌 中 ノ 注 目 ス ベ キ 論 說 記 事

東京物理學雜誌、大正 7 年 2 月號及 1 月號

十一點圓錐曲線

澤山勇三郎氏

三次及ビ四次ノ線狀微分方程式ニ關スルツノ性質

黑須康之介氏

數理雜俎

柳原吉次氏

共軸圓及反形ノ理論ヲ避ケテ一群ノ定理ノ證明

理學士 秋山武太郎氏

哲學雜誌、大正 7 年 1 月號及 2 月號

幾何學ノ論理的基礎

文學士 田邊 元氏

らいぶにつノ哲學ニ就テ

文學士 中島 慎一氏

哲學研究、大正 7 年 2 月號

獨逸唯心論ニ於ケル哲學的認識ノ問題

文學士 田邊 元氏

史學雜誌、大正 6 年 11 月號 12 月號及同 7 年 1 月號

伊能忠敬ガ測地事業ニ成功シタル所以 (梗概)

理學士 大谷 亮吉氏

和算家ノ肖像

三上 義夫氏

日本數學發達ノ由來 (梗概)

同 氏

理學界、大正 6 年 12 月號及同 7 年 1 月號

三角形ノ面積ヲ表ハス式 (58 種)

藤 枝 哲氏

ふえりまノ定理ノ一證明

大神健五郎氏

ふいぎゆれノとなんばノ公式ヲ求ムル簡便法

在薩一理學士

東洋學藝雜誌、大正 7 年 1 月號

世界ノ重ナル國々ニ於ケル幾何學ノ變遷

理學博士 藤澤利喜太郎氏

諸 學 者 ノ 消 息

北米合衆國だ！とますかれ！ぢノはすきん教授 C. N. Haskin ハ半年間ノ賜暇ヲ得タリ、
 べんしるぢゐにあ大學ノぐれん教授 O. E. Glenn 亦半年間ノ賜暇ヲ得、くらいん Dr. J. R.
 Kline 之レニ代レリ。はり！べ！とまん Dr. Harry Bateman ハする！ふぶれつぢニ於ケル
 飛行研究及數理的物理學教授トナレリ。びつばらニ於ケルか！ねぎ！いんすちぢゐ！とニ於
 テも！あへつど Dr. J. C. Morehead ハ數學及畫法幾何學ノ助教トナレリ。いんぢあな大學
 ノみるら！氏 I. L. Miller ハかるせ！ちかれつぢノ教授トナレリ。こらんびあ大學ノしもんづ
 Dr. E. F. Simonds ハいりのいず大學教授トナレリ。へっば！と及り！ど Dr. C. M. Hebbert.
 Dr. F. W. Reed ハいりのいず大學飛行研究所ニ於ケル數學教師トナレリ。かりふゐるにや大
 學ノろいしゆな！教授 A. O. Leuschner ハ合衆國太平洋沿岸航究研究所ノ教授監督者トナレ
 リ。こらんびあ大學ノほうくす教授 H. E. Hawkes ハけつべる學長ノ休職中學長代理ヲ務ムル
 コトトナレリ。ころろど大學ノらいと氏 Dr. G. H. Light ハ數學科副教授トナレリ。ばる
 ちもあしち！かれつぢノのりす教授 S. F. Norris ハ 1917 年 9 月 4 日逝去セリ。

1917 年 3 月ちゆりっひニ於ケル瑞西數學會ニ於テハ巴里 Collège de France 教授あだ
 ま！る氏 J. Hadamard ヲ招聘シタルガ同教授ハ la notion de fonction analytique et les
 equations aux dérivées partielles ナル講演ヲナシタリト。

英吉利ぐらすごう大學ノぎぶそん教授 G. A. Gibson ハえぢんばら Royal Society ノ
 Vice-president トナリ又えぢんばら大學ノうゐとて！か！教教 E. T. Whittaker ハ同會ノ
 Secretary トナレリ。

劔橋大學ノらっせる教授 Bertrand Russell ハ先キニ英國陸軍ノ服役ニ關シ所謂 Conscien-
 tious Objector ナルモノヲ辯護スル小冊子ヲ發行セルガ爲ニ法律ニ依テ所罰セラレ、其ノ爲一
 昨年 1916 年末同大學とりにち！大學評議會 Council of Trinity College ハ同教授ヲ Lecture-
 ship in logic and principles of mathematics ノ職ヨリ逐ヒシカ、本年 (1918) 2 月ノ電報ニ依
 レバ同教授ハ終ニ禁錮六ヶ月ニ處セラレタリトアリ。The foundations of geometry 及 The
 principles of mathematics, vol. 1 ノ著者トシテ、又 Principia mathematica, vols. 1-3 ノ共
 著者トシテ數學界ニ有名ナル同教授ハ一面哲學者ニシテ且政論家ナリ。

京都帝國大學理科大學助教授西内貞吉氏ハ大正七年 (1918) 1 月 28 日同大學教授ニ任ゼラ
 レ、同日數學第二講座擔任ヲ命ゼラレタリ。

本誌第二卷ニ於テ圓理學ニ關スル論文ヲ發表セラレタル徳島縣立德島中學校教諭 武田丑
 太郎氏ハ大正 6 年 (1917) 12 月 19 日逝去セラレタリ。同氏ハ安政 6 年 (1859) 3 月 7 日ノ
 出生ナレバ享年 58 歳ナリ。同氏が徳島縣ノ教育ノ爲ニ將又數學研究獎勵ノ爲ニ盡サレタル效
 蹟ハ頗ル大ナルモノアリ、全國ニ涉リ中等教育界ノ偉動者トシテ有名ナリ。同中學校ハ之ニ報
 ヌル爲ニ未ダ嘗テ類例ナキ校葬ノ式ヲ校庭ニ舉ゲタリ。

On Irreducible Equations Admitting Roots of the Form $\alpha + \rho \cdot e^{i\theta}$, α and ρ Both Rational,

by

AUBREY KEMPNER, Urbana, Ill., U.S.A.

The present paper is a generalization and continuation of a paper by the author on irreducible equations admitting roots of the form $\rho \cdot e^{i\theta}$, ρ rational⁽¹⁾.

While some of the theorems of the present article are straight forward generalizations of theorems contained in the Archiv article (for example theorem I), most of the theorems, in particular all of those dealing with more than one circle of the system S (see "definitions" below) do not have analoga in the simpler case.

Since the complete proofs require hardly more space than would the formal generalizations, the full demonstrations are given in most cases, so that no knowledge of the, at present inaccessible, Archiv article is required.

The results of the paper may be characterized as an extension, for a certain class of equations, of the theorem that, in an equation with real coefficients, the complex roots appear in conjugate pairs, and an examination of the consequences of this extension.

Definitions.

Throughout the paper, unless specifically stated otherwise, *our equations shall have rational coefficients, and reducibility and irreducibility shall be understood to refer to the natural domain $R(1)$.*

We consider in the complex plane the doubly infinite set of circles consisting of all circles of rational radii about each rational point of the

⁽¹⁾ Printed in the Archiv der Mathematik und Physik, Berlin, under the title: „Über irreduzible Gleichungen, die...zulassen." Proof sheets were received in September 1916, but on account of the present disturbed means of communication I do not know in which issue the article is contained. For a brief résumé of the results see Bull. of the Amer. Math. Soc., 1914, vol. 20, p. 183. For some related theorems compare this Journal, vol. 10, 1916, p. 115, and references given there.

axis of reals as centres. The complex number corresponding to any point on the circumference of each of these circles is of the form $a + \rho \cdot e^{i\theta}$, a and ρ both rational, and conversely, all numbers of this type are represented by the points on the circumferences of the circles. In this sense, we shall use the expressions "point on one of the circles" and "a number of the form $a + \rho \cdot e^{i\theta}$, a and ρ both rational," interchangeably.

The whole complex plane is covered everywhere densely by our circumferences, and, as is easily seen, an infinite number of the circles pass through the neighbourhood of any point in the plane. A point of intersection of two circles has an infinite number of circles passing through it.

The circumference of any circle contained in our system we call, for brevity, a circle S , and the system of all circles S , the system S .

Besides the system S we shall also need the system of circles obtained by considering all circumferences of circles whose centres are the rational points on the axis of reals, as before, but whose radii are the square roots of all rational numbers. *This set of circles, which contains S as a partial set, will be called the system C .*

We shall consider chiefly the class of irreducible equations having any complex roots on any circle S . This class of equations consists, as will be seen, of

1. all irreducible equations $f(z)=0$ having any (not necessarily all) complex roots of absolute value unity;
2. all equations derived from 1. by subjecting $f(z)$ to a non-singular transformation $z = \frac{az_1+b}{cz_1+d}$ with real rational coefficients.

In particular, all irreducible quadratic equations with rational coefficients and of negative discriminant belong to our class (§ 2). Obviously 1. contains as a sub-class all cyclotomic equations for n th roots of unity, n prime.

1. Theorem I. *When an irreducible equation has a complex root $a + \rho \cdot e^{i\theta}$, a and ρ^2 both rational (ρ itself may be rational or irrational), the roots of the equation are distributed in the following manner:*

- 1) *Besides $a + \rho \cdot e^{\pm i\theta}$ there may be other pairs of complex roots with the same a and the same ρ , $a + \rho \cdot e^{\pm i\theta_1}$ etc.;*
- 2) *if σ is a real root, $\frac{\rho^2}{\sigma}$ is also a root;*
- 3) *any complex root not contained in 1) may of course be written in the form $a + r \cdot e^{i\phi}$, where a denotes the rational quantity used above.*

Then the four numbers $\alpha + r \cdot e^{\pm i\varphi}$, $\alpha + \frac{\rho^2}{r} e^{\pm i\varphi}$ are all contained among the roots of the equation⁽¹⁾.

Geometrically speaking, our theorem states that if any complex root of an irreducible equation lies on a circle C , then all roots are distributed in the complex plane in such manner that the roots in the interior of the circle are the images of the roots exterior to the circle using the term "image" in the sense in which it is employed in the theory of analytic functions of a complex variable.

Proof: Since $z_1 = \alpha + \rho \cdot e^{i\theta}$ and $z_2 = \alpha + \rho \cdot e^{-i\theta}$ are both roots of our irreducible equation $f(z)=0$, we have

$$(z_1 - \alpha)(z_2 - \alpha) = \rho^2, \quad z_1 z_2 - \alpha(z_1 + z_2) + (\alpha^2 - \rho^2) = 0,$$

$$z_2 = \frac{\alpha z_1 + (\rho^2 - \alpha^2)}{z_1 - \alpha} = \phi(z_1),$$

and

$$\phi(z_2) = \phi\phi(z_1) = z_1;$$

therefore one root z_2 is expressible as a rational function $\phi(z_1)$ with rational coefficients of another root, and iteration of the function $\phi(z_1)$ leads back to the variable z_1 .

Hence, by applying a special case of a well known theorem due to Abel⁽²⁾, all roots may be broken up into pairs so that the members of each pair are related by the same equation; if z_3 is a root different from z_1 and z_2 , then there is another root z_4 such that

$$(z_3 - \alpha)(z_4 - \alpha) = \rho^2.$$

Interpreted in the complex plane, this means that z_3 is the conjugate of the image of z_4 , taken with respect to the circle on whose circumference z_1 and z_2 are situated. In case z_3 is a real root, or a complex root lying on the circle, parts 1. and 2. of our theorem are satisfied.

When z_3 is a complex root not on the circle, then together with z_3

(1) Parts 2) and 3) of Theorem I, and therefore also many of the later theorems hold also when there are no roots on the circumference of any circle C , but when it is known that there is at least one pair of roots z_1, z_2 such that z_1, z_2 are images of each other with respect to a circle C .

(2) Oeuvres, 2nd ed., 1881, vol. 1, p. 478.

For proof of a theorem, according to which an irreducible equation of degree n , between two roots of which a relation exists of the type $z_1 z_2 + A(z_1 + z_2) + B = 0$, where A and B are rational functions of the coefficients of the equation, must be of even degree and is reducible to a quadratic equation by solving an equation of degree $\frac{n}{2}$, and for some related theorems see J. Petersen, *Algebraische Gleichungen*, Kopenhagen, 1878, p. 136.

its conjugate \bar{z}_3 , and together with z_4 its conjugate \bar{z}_4 must be roots of the equation, since $f(z)$ has real coefficients, thus proving Theorem I.

By applying Abel's theorem in its general form, the following theorem, which contains I, is proved in exactly the same way.

Let $f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0$ be an equation with real, but otherwise arbitrary coefficients, irreducible in the domain $R(a_0, a_1, \dots, a_n)$, and assume $f(z) = 0$ to have a root $a + \rho \cdot e^{i\theta}$, a and ρ^2 both rationally expressible in terms of a_0, a_1, \dots, a_n with rational numerical coefficients, then, all roots of $f(z) = 0$ are distributed in the complex plane according to theorem I.

It follows from Theorem I that an irreducible equation having any roots of the form $a + \rho \cdot e^{i\theta}$, a and ρ^2 rational, must be of even degree (excluding as trivial, here and later, linear equations). In particular, when the equation has a root on the unit circle, $|z| = 1$, the equation must be reciprocal,

$$f(z) = a_0 z^{2n} + a_1 z^{2n-1} + \dots + a_{2n-1} z + a_{2n} = 0,$$

and

$$a_0 = +a_{2n}, \quad a_1 = +a_{2n-1}, \quad \text{etc.}$$

By the use of Abel's theorem, it is also seen that when an irreducible equation has two complex roots $a \pm \beta i$, a rational, then all roots are distributed in the complex plane symmetrically with respect to the line parallel to the axis of imaginaries through the real rational point a ; so that, when $a + \gamma$ is any real root (γ then necessarily irrational), $a - \gamma$ is also a root; and when $a + a' + i\beta'$ another complex root ($a' \neq 0$, $\beta' \neq \beta \neq 0$), then the four numbers $a \pm a' \pm i\beta'$ are together roots of the equation⁽¹⁾.

Since symmetry with respect to a straight line is only a special case of inversion with respect to a circle, we see that Theorem I still holds when we arbitrarily admit as degenerate circles in the system C all lines parallel to the axis of imaginaries and at a rational distance from it. However, unless expressly stated, these straight lines will not be taken into account in our theorems.

Combining our results, we may say:

When an irreducible equation has any complex roots on any circle C , or on any straight line parallel to the axis of imaginaries and at a rational distance from it, this circle or this straight line shares the fundamental property of the axis of reals, that the roots are distributed on both sides in

(1) See also the Archiv article already referred to.

such manner that the roots on one side are the images of the roots on the other side.

2. Quadratic irrationalities.

Theorem II. The (in the complex plane everywhere dense) points of intersection of two circles S are exactly the roots of the system of irreducible equations $az^2+bz+c=0$, where a, b, c are any rational numbers and $b^2-4ac < 0$.

Proof. a) We first show that every point of intersection of two circles C is a quadratic irrationality. The points of intersection of two circles S then are of course also quadratic irrationalities. The situation is illustrated by Fig. 1. Let P be the point of intersection,

$$P = a_1 + \rho_1 \cdot e^{i\theta_1} = a_2 + \rho_2 \cdot e^{i\theta_2},$$

$a_1, a_2, \rho_1^2, \rho_2^2$ all rational.

We must show that the two complex numbers $z_1 = x + iy$ and $z_2 = x - iy$ are the roots of a quadratic equation with rational coefficients.

We shall have

$$(a_2 - x)^2 + y^2 = \rho_2^2; \quad y^2 + (x - a_1)^2 = \rho_1^2;$$

$$2x(a_1 - a_2) = \rho_2^2 - \rho_1^2 + a_1^2 - a_2^2,$$

and x is rational, while $y = \sqrt{\rho_1^2 - (x - a_1)^2}$ is the square root of a rational number. But $z = a + i \cdot \sqrt{b}$, where a and b both rational, is a quadratic irrationality.

b) To show that the roots of an irreducible equation $az^2+bz+c=0$, a, b, c rational, $b^2-4ac > 0$, are points of intersection of two circles S , we proceed as follows.

From $z = x \pm iy = \frac{1}{2a}(-b \pm i\sqrt{4ac-b^2})$ we see that $y = \frac{1}{2a}\sqrt{R}$,

where x and $R = 4ac - b^2 > 0$ are both rational. We have to show that we may choose in Fig. 1 the four lengths a_1, a_2, ρ_1, ρ_2 all rational. Assuming $\rho_1 = \rho_2$, we shall have

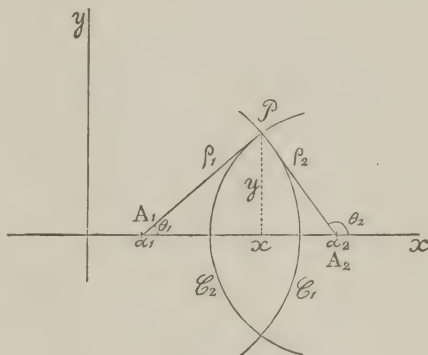


Fig. 1.

$$x = \frac{1}{2}(a_1 + a_2), \quad a_2 - x = x - a_1,$$

and
$$\frac{R}{4a^2} = y^2 = \rho_2^2 - (a_2 - x)^2 = (\rho_2 + a_2 - x)(\rho_2 - a_2 + x).$$

Let $2a(\rho_2 - a_2 + x) = R_1$, where R_1 is any positive rational number, then $2a(\rho_2 + a_2 - x) = R_2$, where $R_2 = \frac{R}{R_1}$ is also rational, and $\rho_2 = \rho_1 = \frac{R_1 + R_2}{4a}$.

The quantities a_1 and a_2 are then also rational (from $2a(\rho_2 - a_2 + x) = R_1$).

Theorem II is thus proved, and we may even choose the radii of our circles to be equal. It is easily seen that an infinite number of circles pass through each point of intersection of two circles S .

While the system C yields also all real quadratic irrationalities, as the points of intersection of the circles with the axis of reals, the set S does not yield the real irrationalities.

Since all points of intersection of circles S are at the same time points of intersection of circles C , we see from our theorem that the everywhere dense set of points of intersection of the circles S is identical with the set of points of intersection of the circles C .

As a corollary to Theorem II, it follows that while any complex quadratic irrationality may be represented in an infinite number of ways in the form $\alpha + \rho \cdot e^{i\theta}$, α and ρ both rational, any root of an irreducible equation of degree $n > 2$ can be represented in only one way in the form $\alpha + \rho \cdot e^{i\theta(1)}$, if it can be at all so represented; consequently, for $n > 2$, $\alpha_1 + \rho_1 \cdot e^{i\theta_1} = \alpha_2 + \rho_2 \cdot e^{i\theta_2}$, $\alpha_1, \rho_1, \alpha_2, \rho_2$ all rational, implies $\alpha_1 = \alpha_2$, $\rho_1 = \pm \rho_2$.

3. Roots on more than one circle S .

Theorem III. An irreducible equation cannot have roots on more than one circle S unless the circles intersect.

Proof: Assume two circles S_1 and S_2 , exterior to each other, and assume $2\mu_1$ roots on S_1 , $2\mu_2$ roots on S_2 ⁽²⁾. Let $n = 2m$ be the degree of our equation, then there must be, by Theorem I, exactly $m - \mu_1$ roots inside S_1 , and as many outside. For the same reason, there are $m - \mu_2$ roots inside S_2 . Let $\lambda \geq 0$ be the number of roots exterior to both circles, then we shall have

$$m - \mu_1 = \lambda + 2\mu_2 + (m - \mu_2), \quad \lambda = -\mu_1 - \mu_2.$$

(1) Not counting as distinct the two representations $\alpha + \rho \cdot e^{i\theta}$, $\alpha - \rho \cdot e^{i(\pi + \theta)}$.

(2) When a real root lies on S_1 or on S_2 , the equation is linear

This is impossible, since μ_1, μ_2 are positive.

When one circle contains the other, say S_2 in S_1 , we may transform S_1 into the unit-circle about the origin by a transformation $z_1 = \gamma z + \delta$ with real rational coefficients, and by a transformation $z_2 = z^{-1}$ carry the circle into which S_2 is transformed by $z_1 = \gamma z + \delta$, and which now lies inside the unit-circle, into a circle lying outside the unit-circle. Since neither the degree of the equation nor its irreducibility are affected by the transformations, we are reduced to the case just treated.

The proof requires no modification in case the circles touch each other on the axis of reals, internally or externally.

When the circles intersect, we know that the points of intersection are quadratic irrationalities. Excluding this case, we have to answer the question: Can roots of an irreducible equation of degree $n > 2$ lie on both of two intersecting circles? The answer is given by

Theorem IV. On every circle S there are roots of irreducible equations of degree $n > 2$, some of whose other roots lie on an intersecting circle.

Proof: It is only necessary to prove the theorem for any particular circle S , since all other circles S are derivable from any one by a linear transformation with real rational coefficients, and the transformation does not affect the irreducibility of the equation. It follows from Theorem I that the lowest degree for which an irreducible equation may have complex roots on two distinct circles, excepting $n=2$, is $n=6$. To form such an equation, we choose (see Fig. 2) the unit-circle about the origin as centre for one of our circles, S_1 , and the circle of radius unity about the point $(1, 0)$ for the second circle, S_2 . Draw through the origin any straight line which intersects in the upper half plane the common chord of the two circles, in the point ϵ_1 , and let ϵ_2 be the point of intersection in the upper half plane of the line with S_2 . Draw also the line through $(1, 0)$ to ϵ_1 , and call ϵ_3 the point of intersection of this line with S_1 ; finally, let $\epsilon_{-1}, \epsilon_{-2}, \epsilon_{-3}$ respectively be the conjugate points to $\epsilon_1, \epsilon_2, \epsilon_3$.

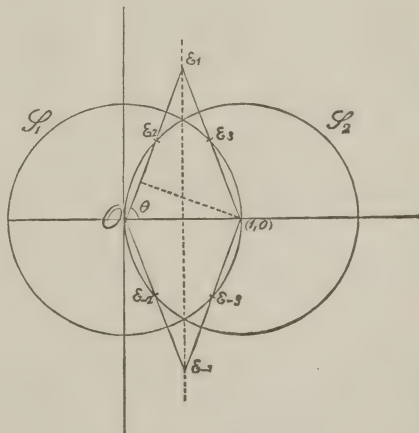


Fig. 2.

Let $\varepsilon_2 = r \cdot e^{i\theta}$, $r > 0$, so that r is the distance from the origin to ε_2 . From elementary geometry we see that $|\varepsilon_1 \cdot \varepsilon_2| = 1$, and obtain, making use of the obvious relation $r = 2 \cos \theta$, the six equations

$$\varepsilon_{\pm 1} = r^{-1} (\cos \theta \pm i \sin \theta) = (2r)^{-1} \cdot (r \pm i \sqrt{4 - r^2}),$$

$$\varepsilon_{\pm 2} = r \cdot (\cos \theta \pm i \sin \theta) = \frac{1}{2} r \cdot (r \pm i \sqrt{4 - r^2}),$$

$$\varepsilon_{\pm 3} = 1 - \varepsilon_{\mp 2} = 1 - r \cos \theta \pm i r \sin \theta = \frac{1}{2} (2 - r^2 \pm i r \sqrt{4 - r^2}).$$

The points $\varepsilon_1, \varepsilon_{-1}$ lie on the line parallel to the axis of imaginaries through the real rational point $\frac{1}{2}$, and therefore, by § 1, if $\varepsilon_1, \varepsilon_{-1}$ are to be roots of an irreducible equation, the other roots must be symmetrically distributed in the complex plane with reference to this line. In particular, if any one of the four values $\varepsilon_2, \varepsilon_3, \varepsilon_{-2}, \varepsilon_{-3}$ is a root of the equation, all of them must be roots. We shall show how to choose r so that the six quantities $\varepsilon_1, \dots, \varepsilon_{-3}$ are the roots of an irreducible equation with rational coefficients.

From the relations (see Fig. 2 and the expressions for the ε derived above)

$$\varepsilon_1 + \varepsilon_{-1} = 1, \quad \varepsilon_1 \cdot \varepsilon_{-1} = r^{-2},$$

$$\varepsilon_2 + \varepsilon_{-2} = 2r \cos \theta = r^2, \quad \varepsilon_2 \cdot \varepsilon_{-2} = r^2,$$

$$\varepsilon_3 + \varepsilon_{-3} = 2 - r^2, \quad \varepsilon_3 \cdot \varepsilon_{-3} = 1,$$

we see that the quadratic equation whose roots are $\varepsilon_1, \varepsilon_{-1}$ is

$$z^2 - z + r^{-2} = 0,$$

while the equation with roots $\varepsilon_2, \varepsilon_{-2}$ is

$$z^2 - r^2 z + r^2 = 0,$$

and the equation with roots $\varepsilon_3, \varepsilon_{-3}$ is

$$z^2 - (2 - r^2)z + 1 = 0.$$

By imposing on r the first restriction that r^2 shall be irrational, we ensure that none of these three equations has rational coefficients. But if our sextic equation were reducible in $R(1)$, at least one of these equations would have rational coefficients. Therefore we only have to show that r may also be chosen so that $\varepsilon_1, \dots, \varepsilon_{-3}$ will be the roots of an equation with rational coefficients.

The equation with roots $\varepsilon_1, \dots, \varepsilon_{-3}$ is

$$z^6 - 3z^5 + r \cdot z^4 - z^3 (2r - 5) + r \cdot z^2 - 3z + 1 = 0,$$

where $\gamma = r^{-2} + 3 + 3r^2 - r^4$, and we must select r so that

1. r^2 is irrational,
2. $r^{-2} + 3 + 3r^2 - r^4$ is rational.

writing $r^2 = x$, we have $x^{-1} + 3 + 3x - x^2 = \gamma$, and we choose, if possible, for γ an integral value which will make $1 + (3 - \gamma) \cdot x + 3x^2 - x^3 = 0$ irreducible, and so that at least one root x is real, positive, and between zero and four (because $0 < r < 2$). This happens, for example, for $\gamma = 7$, $x^3 - 3x^2 + 4x - 1 = 0$. Condition 1. is of course also satisfied; therefore the equation with roots $\varepsilon_1, \varepsilon_{-1}, \varepsilon_2, \varepsilon_{-2}, \varepsilon_3, \varepsilon_{-3}$, where r is to be taken equal to the positive square root of the real root between zero and four of the equation $x^3 - 3x^2 + 4x - 1 = 0$, has rational coefficients, is irreducible, and has roots on both circles S_1 and S_2 .

From Theorems III and IV follows: An irreducible equation $f(z) = 0$ can have two roots $\alpha_1 + \rho_1 \cdot e^{i\theta_1}$, $\alpha_2 + \rho_2 \cdot e^{i\theta_2}$, $\alpha_1, \alpha_2, \rho_1, \rho_2$ all rational and $\alpha_1 \neq \alpha_2$, only when

$$|\rho_1 - \rho_2| < |\alpha_1 - \alpha_2| < \rho_1 + \rho_2.$$

4. Determination of α and ρ .

Theorem V. When an irreducible equation $f(z) = 0$ has complex roots of the form $z = \alpha + \rho \cdot e^{i\theta}$, α and ρ rational, the quantities α and ρ can be found by a finite number of rational operations and the extraction of a square root.

Our theorem implies that when an irreducible equation has a root on any circle S , this circle can be constructed by ruler and compasses.

Proof: Let $f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0$, $f(z)$ irreducible, and $z = \alpha + \rho \cdot z_1$, $z_1 = e^{i\theta}$, a complex root, where α and ρ are rational. Then

$$\begin{aligned} f(z) &= f(\alpha + \rho z_1) = \varphi(z_1) \\ &= f(\alpha) + \frac{f'(\alpha)}{1!} \rho z_1 + \dots + \frac{f^{(n-1)}(\alpha)}{(n-1)!} \rho^{n-1} z_1^{n-1} + \frac{f^{(n)}(\alpha)}{n!} \rho^n z_1^n \\ &= c_0 + c_1 z_1 + \dots + c_{n-1} z_1^{n-1} + c_n z_1^n, \end{aligned}$$

and $\varphi(z_1) = 0$ is of degree n and irreducible. But since $\varphi(z_1) = 0$ has a root $z_1 = e^{i\theta}$, it must, by Theorem I, be (of even degree and) reciprocal, $c_0 = +c_n$, $c_1 = +c_{n-1}, \dots$. Writing for convenience $f(\alpha) = f$, $f^{(\mu)}(\alpha) = f_\mu$,

and $[f^{(\mu)}(a)]^\lambda = f_\mu^\lambda$, the equations $c_0 = c_n$ and $c_1 = c_{n-1}$ yield the relations⁽¹⁾:

$$\rho^n = \frac{n! f}{f_n} = \frac{f}{a_0},$$

$$\rho^{n-2} = \frac{(n-1)! f_1}{f_{n-1}} = \frac{f_1}{n a_0 a + a_1}.$$

Therefore

$$\rho^2 = \frac{n f f_{n-1}}{f_1 f_n} = \frac{(n a_0 a + a_1) f}{a_0 f_1},$$

and

$$\rho^n = (\rho^2)^{\frac{n}{2}} = \frac{(n a_0 a + a_1)^{\frac{n}{2}} f^{\frac{n}{2}}}{a_0^{\frac{n}{2}} f_1^{\frac{n}{2}}} = \frac{f}{a_0},$$

$$(1) \quad (n a_0 a + a_1)^{\frac{n}{2}} f^{\frac{n}{2}-1} = a_0^{\frac{n}{2}-1} f_1^{\frac{n}{2}},$$

which may be written, more symmetrically,

$$n! f^{\frac{n}{2}} f_n^{\frac{n}{2}-1} = n^{\frac{n}{2}} f^{\frac{n}{2}-1} f_{n-1}^{\frac{n}{2}}.$$

This equation with rational coefficients must be satisfied by our α . To find the rational roots of such an equation, only a finite number of rational operations are required. It is readily seen that the equation does not reduce to an identity, since one side of the equation contains the coefficient a_n , while the other does not. However, the apparent degree of the equation (1) in α , that is $\frac{1}{2}(n^2 - n)$, is not the actual degree of the equation, since, besides the coefficient of the highest power of α , a considerable number of leading coefficients seem to vanish identically for any given polynomial $f(z)$ of even degree.

Only for $n=2$ does the method break down. This case, however, has been fully treated in § 2. Once α is determined, the corresponding ρ is derived either from $\rho^n = \frac{f(\alpha)}{a_0}$, or (by extraction of a square root)

$$\text{from } \rho^2 = \frac{(n a_0 a + a_1) f(\alpha)}{a_0 f(\alpha)}.$$

$$\text{Example: } f(z) = 16z^4 + 72z^3 + 96z^2 + 60z + 15 = 0.$$

$$\text{Equation } (n a_0 a + a_1)^{\frac{n}{2}} f^{\frac{n}{2}-1} = a_0^{\frac{n}{2}-1} f_1^{\frac{n}{2}} \text{ gives}$$

$$(64\alpha + 72)^2 (16\alpha^4 + 72\alpha^3 + 96\alpha^2 + 60\alpha + 15) = 16 (64\alpha^2 + 216\alpha^2 + 192\alpha + 60)^2,$$

(1) These equations are the first two members of a set which may be expressed by the formula $\rho^{2\lambda} = \frac{(m+\lambda)! f_{m-\lambda}}{(m-\lambda)! f_{m+\lambda}}$, where $n=2m$ and $\lambda=1, 2, \dots, m-1$.

which is satisfied by the rational value $\alpha = -\frac{1}{2}$. The corresponding value of ρ is $\frac{1}{2}$. As a matter of fact, the given irreducible equation has two real roots $\frac{1}{8}\{-\sqrt{21}-9 \pm \sqrt{30+10\sqrt{21}}\}$, and on the circle $-\frac{1}{2} + \frac{1}{2}e^{i\theta}$ two complex roots, $\frac{1}{8}\{+\sqrt{21}-9 \pm \sqrt{30-10\sqrt{21}}\}$.

5. Necessary and sufficient conditions for the existence of complex roots on circles S .

We still have to find necessary and sufficient conditions for the existence of roots $\alpha + \rho \cdot e^{i\theta}$, α and ρ rational, because in § 4 nothing has been said concerning θ , and therefore the equations for α and ρ have only the character of necessary conditions.

We make use of a known theorem which gives necessary and sufficient conditions for the existence of complex roots of the form $e^{i\theta}$ of any equation with real coefficients⁽¹⁾, and obtain without difficulty the following theorem which, while awkward in application, theoretically completely solves the problem:

Theorem VI. An irreducible equation $f(z)=0$ has complex roots $\alpha + \rho \cdot e^{i\theta}$, α and ρ rational, when and only when both of the following conditions are satisfied:

1. The system of equations for α , ρ

$$\rho^n \cdot a_0 = f(\alpha),$$

$$\rho^{n-2} \cdot (n a_0 \alpha + a_1) = f'(\alpha)$$

has a rational solution (α, ρ) ;

2. letting $z = \alpha + \rho \cdot z_1$ and forming the expressions

$$g(z_1) = f(\alpha + \rho \cdot z_1),$$

$$\phi(z_1) = (z_1^2 + 1)^n \cdot g\left(\frac{z_1 + i}{z_1 - i}\right) \cdot g\left(\frac{z_1 - i}{z_1 + i}\right)^{(2)} = \phi(z_1^2) = \phi(u),$$

then the equation $\phi(u)=0$ must admit at least one positive root.

To apply the test, we should find all rational solutions (α_1, ρ_1) , $(\alpha_2, \rho_2), \dots$ of 1., and then apply 2. to each of these solutions separately.

(1) Kempner, this Journal, Vol. 10, 1916, p. 115.

(2) The operations indicated in the next two steps are justified because $\phi(z_1)$ contains only even powers of z_1 ; it is, in fact, a polynomial in z^2 with real coefficients. Compare l. c.

6. Applications.

The equation $a_0 \rho^n = f(\alpha)$ of § 4 can be stated as follows:

Theorem VII. When an irreducible equation $f(z)=0$ has a complex root $\alpha + \rho \cdot e^{i\theta}$, α and ρ rational, then the diophantine equation

$$a_0 y^n = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

has a rational solution (x and y rational).

From this, some interesting corollaries may be deduced:

VII.^a When α and ρ are both integers, then

$$a_0 y^n = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

has an integral solution (x, y).

The circles in the complex plane in this case consist of all circles of integral radii about all integral points of the axis of reals as centres.

VII.^b If $f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n = 0$ has a complex root on any circle of rational radius touching the axis of imaginaries at the origin, then the equation obtained by omitting the highest power of z ,

$$a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n = 0$$

has a rational root.

The conditions of *VII.^b* are satisfied, for example, by the equation (see § 4)

$$16z^4 + 72z^3 + 96z^2 + 60z + 15 = 0.$$

The equation

$$72z^3 + 96z^2 + 60z + 15 = 0$$

admits the root $z = -\frac{1}{2}$.

Letting our system of circles consist of all circles of radius unity about all rational points of the axis of reals as centres we find:

VII.^c When $f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n = 0$ has a root on any of these circles, the equation

$$a_0 x^n + a_1 x^{n-1} + \dots + a_n = a_0$$

has a rational root; or, as we may say instead,

$$a_0 u^n + a_1 u^{n-1} v + \dots + a_n v^n = a_0 v^n$$

has an integral solution (u, v).

As a second application we mention that every irreducible equation of degree $n < 10$ with a root on any circle S is completely solvable by radicals.

This is obvious, when one remembers that α and ρ may be deter-

mined, and that $\varphi(z_1) = f(a + \rho z_1) = 0$ is, as an equation in z_1 , of even degree, and reciprocal. Its degree is therefore not greater than eight. But such equations are always solvable by radicals.

As a last application, consider the figure obtained in the following manner: Given the set of circles of radius one-half with centres on the axis of reals at $0, \pm 1, \pm 2, \dots$ in inf., and also the system of straight lines parallel to the axis of imaginaries which are tangent to these circles. Reflect in each circle (by reciprocal radii) all given circles and straight lines, and continue this indefinitely with all circles. In this manner, in the limit, a figure is obtained, consisting of a set of circles such that, through every rational point on the axis of reals, circles of rational radii (and therefore with rational centres) pass; no two circles of the whole system intersect⁽¹⁾.

Therefore the conditions of Theorem III are satisfied, and we see that *an irreducible equation cannot have roots on more than one circle of this figure.*

When for an irreducible equation it happens to be known that *all* roots lie on a circle S , the exact location of the roots on the circle may frequently be determined by transforming the circle into the circle of radius unity about the origin and making use of the following theorem⁽²⁾. "When all roots of an irreducible equation with integral coefficients are of absolute value unity, and the coefficient of the highest power of the unknown is unity, then all roots are roots of unity."

(1) This figure, at least as far as it lies in the upper half-plane, is one of F. Klein's famous "Modul" figures and is well known in the theory of elliptic modular functions, continued fractions, and reduction of binary quadratic forms with positive discriminants. Compare for example

F. Klein, *Elliptische Modulfunktionen*, 1890, Vol. I, p. 273, or *Mathematische Annalen*, Vol. XIV, p. 119;

G. Humbert, *Lieuville Journal*, 1916, p. 104;

H. St. Smith, *Collected Papers*, Vol. II, p. 224.

(2) Kronecker, *Werke*. Vol. I, p. 107.

Note on Dr. Muir's Paper on "A Theorem Including Cayley's on Zero-Axial Skew Determinants of even Order"⁽¹⁾,

by

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The principal object of this note is to give a very simple proof of Dr. Muir's extension of Cayley's theorem connecting a skew-symmetric determinant with a Pfaffian.

Starting with the determinant

$$\Delta_5 \equiv \begin{vmatrix} 1 & -x & -x & -x & -x \\ 1 & 0 & a-x & b-x & c-x \\ 1 & x-a & 0 & d-x & e-x \\ 1 & x-b & x-d & 0 & f-x \\ 1 & x-c & x-e & x-f & 0 \end{vmatrix},$$

we find, by adding x times the first column to each of the other columns, that

$$\begin{aligned} \Delta_5 &\equiv \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & x & a & b & c \\ 1 & 2x-a & x & d & e \\ 1 & 2x-b & 2x-d & x & f \\ 1 & 2x-c & 2x-e & 2x-f & x \end{vmatrix} \\ &= \begin{vmatrix} x & a & b & c \\ 2x-a & x & d & e \\ 2x-b & 2x-d & x & f \\ 2x-c & 2x-e & 2x-f & x \end{vmatrix} \end{aligned}$$

$=M_4$ say, which is Dr. Muir's form.

⁽¹⁾ This Journal, Vol. 11, 1917, p. 205.

Again

$$\Delta_5 \equiv \begin{vmatrix} 0 & -x & -x & -x & -x \\ 1 & 0 & a-x & b-x & c-x \\ 1 & x-a & 0 & d-x & e-x \\ 1 & x-b & x-d & 0 & f-x \\ 1 & x-c & x-e & x-f & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & a-x & b-x & c-x \\ x-a & 0 & d-x & e-x \\ x-b & x-d & 0 & f-x \\ x-c & x-e & x-f & 0 \end{vmatrix}$$

$= S_5 + P_4^2$ say, where P_4 is the Pfaffian square root of the skew-symmetric determinant of order four.

Since S_5 is a skew-symmetric determinant of odd order and therefore vanishes we have

$$\Delta_5 = P_4^2$$

and therefore

$$M_4 = P_4^2.$$

The method used here obviously applies to any even order.

The simplest way to get the corresponding relation for the case of odd order is to do as Dr. Muir suggests and put $c=e=f=0$ in the case of the next higher even order and get

$$\Delta_5 = x^2 \cdot \begin{vmatrix} 0 & a-x & b-x & -1 \\ x-a & 0 & d-x & -1 \\ x-b & x-d & 0 & -1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$= x \cdot M_3.$$

If we denote the skew-symmetric determinant

$$\begin{vmatrix} 0 & a & b & \dots \\ -a & 0 & c & \dots \\ -b & -c & 0 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

by S_{2n} or S_{2n-1} according as the order is even or odd; and the bordered determinant

$$\begin{vmatrix} 1 & -x & -x & -x \dots\dots \\ 1 & 0 & a & b \dots\dots \\ 1 & -a & 0 & c \dots\dots \\ 1 & -b & -c & 0 \dots\dots \\ \dots\dots\dots\dots\dots\dots \end{vmatrix}$$

by Δ_{2n+1} or Δ_{2n} according as the order is even or odd, then from what precedes we see that the following statements may be made:

I. $\Delta_{2n+1} = S_{2n}$, or the determinant S_{2n} is not altered by being bordered in this way.

II. A skew-symmetric determinant of even order is not altered by adding the same number to each element.

III. $\Delta_{2n} = x \cdot S_{2n}'$, where

$$S_{2n}' = \begin{vmatrix} 0 & -1 & -1 & -1 \dots\dots \\ 1 & 0 & a & b \dots\dots \\ 1 & -a & 0 & c \dots\dots \\ 1 & -b & -c & 0 \dots\dots \\ \dots\dots\dots\dots\dots\dots \end{vmatrix}_{2n}$$

October 1917.

Sur une propriété de la courbure de certaines courbes associées au triangle,

par

R. GOORMAGHTIGH, Londres.

1. Soient $A_1 A_2 A_3$ le triangle de référence et P un point de coordonnées barycentriques μ_1, μ_2, μ_3 . La transformation isotomique fait correspondre au point P le point P' de coordonnées barycentriques relatives $\frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{1}{\mu_3}$. Parmi les cubiques qui se transforment en elles-mêmes par cette transformation, considérons celles

$$\Sigma a_1 \mu_1 (\mu_2^2 - \mu_3^2) = 0, \quad (1)$$

qui sont les lieux des points P tels que la droite qui joint P à son conjugué isotomique P' passe par un point fixe $Q(a_1, a_2, a_3)$. D'après des propriétés bien connues, la cubique correspondant aux coefficients a_1, a_2, a_3 passe par les points de coordonnées barycentriques (a_1, a_2, a_3) et $\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}\right)$ ainsi que par le centre de gravité G du triangle où elle touche la droite GQ . Nous nous proposons de démontrer d'abord, au sujet des cubiques (1), cette propriété que nous croyons nouvelle :

Le centre de courbure de l'une quelconque des cubiques considérées au centre de gravité du triangle est le même que celui de la courbe anharmonique d'Halphen qui la touche en ce point.

Dérivons l'équation (1) par rapport à l'arc s de la courbe ; nous aurons

$$\Sigma a_1 (\mu_2^2 - \mu_3^2) \frac{d\mu_1}{ds} + 2 \Sigma a_1 \mu_1 \left(\mu_2 \frac{d\mu_2}{ds} - \mu_3 \frac{d\mu_3}{ds} \right) = 0.$$

En dérivant encore une fois cette équation par rapport à s , on a

$$\begin{aligned} \Sigma a_1 (\mu_2^2 - \mu_3^2) \frac{d^2 \mu_1}{ds^2} + 4 \Sigma a_1 \left(\mu_2 \frac{d\mu_2}{ds} - \mu_3 \frac{d\mu_3}{ds} \right) \frac{d\mu_1}{ds} \\ + 2 \Sigma a_1 \mu_1 \left[\left(\frac{d\mu_2}{ds} \right)^2 - \left(\frac{d\mu_3}{ds} \right)^2 \right] + 2 \Sigma a_1 \mu_1 \left(\mu_2 \frac{d^2 \mu_2}{ds^2} - \mu_3 \frac{d^2 \mu_3}{ds^2} \right) = 0. \end{aligned}$$

Si l'on considère comme point (μ_1, μ_2, μ_3) de la cubique le centre de

gravité, on a $\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$ et les deux relations qui précèdent s'écrivent plus simplement

$$\Sigma a_1 \left(\frac{d\mu_2}{ds} - \frac{d\mu_3}{ds} \right) = 0, \quad (2)$$

$$2 \Sigma a_1 \left(\frac{d\mu_2}{ds} - \frac{d\mu_3}{ds} \right) \frac{d\mu_1}{ds} + \frac{1}{3} \Sigma a_1 \left(\frac{d^2\mu_2}{ds^2} - \frac{d^2\mu_3}{ds^2} \right) + \Sigma a_1 \left[\left(\frac{d\mu_2}{ds} \right)^2 - \left(\frac{d\mu_1}{ds} \right)^2 \right] = 0. \quad (3)$$

Soient x_1, x_2, x_3 et y_1, y_2, y_3 les distances, prises avec leurs signes, des sommets A_1, A_2, A_3 du triangle à la normale et à la tangente à la cubique (1) au centre de gravité; d'après les formules de la Géométrie intrinsèque⁽¹⁾, on aura, en désignant par ρ le rayon de courbure de la cubique (1) en G et par a^2 l'aire du triangle $A_1 A_2 A_3$,

$$\frac{d\mu_1}{ds} = \frac{y_2 - y_3}{a^2}, \quad \frac{d\mu_2}{ds} = \frac{y_3 - y_1}{a^2}, \quad \frac{d\mu_3}{ds} = \frac{y_1 - y_2}{a^2}, \quad (4)$$

$$\frac{d^2\mu_1}{ds^2} = -\frac{x_2 - x_3}{a^2\rho}, \quad \frac{d^2\mu_2}{ds^2} = -\frac{x_3 - x_1}{a^2\rho}, \quad \frac{d^2\mu_3}{ds^2} = -\frac{x_1 - x_2}{a^2\rho}. \quad (5)$$

L'équation (2) s'écrira donc, puisque $\Sigma y_1 = 0$,

$$\Sigma a_1 y_1 = 0; \quad (6)$$

on a par conséquent

$$\frac{y_1}{a_2 - a_3} = \frac{y_2}{a_3 - a_1} = \frac{y_3}{a_1 - a_2}. \quad (7)$$

L'équation (3) s'écrit ensuite, en tenant compte de la relation $\Sigma x_1 = \Sigma y_1 = 0$,

$$\rho = \frac{a^2 \Sigma a_1 x_1}{3 \Sigma a_1 y_1 (y_2 - y_3)}.$$

Cette expression peut encore s'écrire, en égard aux égalités (7),

$$\rho = \frac{a^2 \Sigma a_1 (y_2 - y_3)}{3 \Sigma a_1 y_1 (y_2 - y_3)^2} = -\frac{a^4}{27 y_1 y_2 y_3}.$$

On sait que si y_1, y_2, y_3 désignent les distances des sommets A_1, A_2, A_3 à la tangente au point (μ_1, μ_2, μ_3) de la courbe triangulaire symétrique $\Sigma \beta_1 \mu_1^n = 0$, le rayon de courbure de cette courbe en ce point a pour expression⁽²⁾

(1) Cesàro, *Vorlesungen über natürliche Geometrie*, p. 124.

(2) Cesàro, *Vorlesungen über natürliche Geometrie*, p. 129.

$$\frac{a^4}{n-1} \frac{\mu_1 \mu_2 \mu_3}{y_1 y_2 y_3};$$

dans le cas limite où n est nul, la courbe triangulaire devient une courbe anharmonique $\mu_1 c_1 \mu_2 c_2 \mu_3 c_3 = \text{const}$, où $c_1 + c_2 + c_3 = 0$. La valeur du rayon de courbure de la cubique (1) au point G est donc égale en grandeur et en signe à celle du rayon de courbure de la courbe anharmonique qui touche la cubique en G , ce qui démontre le théorème énoncé ci-dessus.

2. *Lieu géométrique des centres de courbure des cubiques (1) au centre de gravité du triangle de référence.*—Il résulte des considérations qui précèdent que le centre de courbure de la conique circonscrite au triangle $A_1 A_2 A_3$ qui touche la cubique (1) en G est le milieu du rayon de courbure de la cubique en ce point. Par conséquent la recherche du lieu géométrique des centres de courbure de toutes les cubiques (1) au point G revient à celle du lieu des centres de courbure d'un faisceau de coniques en l'un de ses points de base.

Considérons donc les coniques passant par quatre points A, B, C, D et déterminons le lieu des centres de courbure de ces coniques correspondant au point de base A . Une droite menée par A rencontre en p et q les droites BD et CD , les perpendiculaires élevées en A sur AB et AC coupent en q' et p' les perpendiculaires élevées en q et p sur pq ; si R désigne l'intersection de $p'q'$ avec la perpendiculaire élevée en A sur pq , le milieu ω de AR est le centre de courbure en A de la conique du faisceau qui touche pq en A (¹).

On déduit aisément de cette construction que, lorsque la droite pq varie, le lieu géométrique du point ω est une cubique dont les asymptotes sont perpendiculaires à AB, AC, AD . En particulier, quand les angles BAC, CAD, DAB sont de 120° , la courbe obtenue est une trisectrice de de Longchamps(²), courbe polaire réciproque de l'hypocycloïde à trois rebroussements.

On a donc le théorème suivant :

Le lieu géométrique des centres de courbure des cubiques (1) au centre de gravité du triangle est une cubique dont les asymptotes sont perpendiculaires aux médianes; quand le triangle est équilatéral le lieu géométrique considéré est une trisectrice de de Longchamps.

3. *Généralisation.*—L'équation (1) est un cas particulier de l'équation

$$\Sigma \mu_1 f(\mu_1) [\varphi(\mu_2) - \varphi(\mu_3)] = 0; \quad (8)$$

(¹) P. Serret, *Géométrie de direction*.

(²) G. Loria, *Spezielle ebene Kurven*, t. I, p. 92.

nous allons démontrer pour les courbes représentées par cette équation, un théorème qui généralise la proposition établie au paragraphe 1.

En dérivant deux fois l'équation (8) par rapport à s , on a successivement

$$\Sigma a_1 f'(\mu_1) [\varphi(\mu_2) - \varphi(\mu_3)] \frac{d\mu_1}{ds} + \Sigma a_1 f(\mu_1) \left[\varphi'(\mu_2) \frac{d\mu_2}{ds} - \varphi'(\mu_3) \frac{d\mu_3}{ds} \right] = 0, \quad (9)$$

$$\begin{aligned} & \Sigma a_1 f''(\mu_1) [\varphi(\mu_2) - \varphi(\mu_3)] \frac{d\mu_1}{ds} + \Sigma a_1 f'(\mu_1) [\varphi(\mu_2) - \varphi(\mu_3)] \frac{d^2\mu_1}{ds^2} \\ & + 2 \Sigma a_1 f'(\mu_1) \left[\varphi'(\mu_2) \frac{d\mu_2}{ds} - \varphi'(\mu_3) \frac{d\mu_3}{ds} \right] \frac{d\mu_1}{ds} \\ & + \Sigma a_1 f(\mu_1) \left[\varphi''(\mu_2) \left(\frac{d\mu_2}{ds} \right)^2 - \varphi''(\mu_3) \left(\frac{d\mu_3}{ds} \right)^2 \right. \\ & \left. + \varphi'(\mu_2) \frac{d^2\mu_2}{ds^2} - \varphi'(\mu_3) \frac{d^2\mu_3}{ds^2} \right] = 0. \end{aligned} \quad (10)$$

Si l'on fait dans l'équation (9) $\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$ et si l'on tient compte des relations (4) ainsi que de la relation $\Sigma y_i = 0$, on trouve

$$\Sigma a_1 y_i = 0,$$

relation identique à (6). On a donc cette proposition qui généralise une propriété bien connue des cubiques (1) :

Quelles que soient les fonctions données f et φ , la tangente à la courbe (8) au centre de gravité du triangle passe par le point dont les coordonnées barycentriques relatives sont a_1, a_2, a_3 .

En posant de même dans l'équation (10) $\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}$, on trouve, en désignant encore par ρ le rayon de courbure de la courbe (8) au point G ,

$$\begin{aligned} \rho &= \frac{f\left(\frac{1}{3}\right)\varphi'\left(\frac{1}{3}\right)\alpha^2}{\left[2f'\left(\frac{1}{3}\right)\varphi'\left(\frac{1}{3}\right) - f\left(\frac{1}{3}\right)\varphi''\left(\frac{1}{3}\right)\right]} \times \frac{\Sigma a_1 x_1}{\Sigma a_1 y_1(y_2 - y_3)} \\ &= \frac{f\left(\frac{1}{3}\right)\varphi'\left(\frac{1}{3}\right)\alpha^4}{9\left[f\left(\frac{1}{3}\right)\varphi''\left(\frac{1}{3}\right) - 2f'\left(\frac{1}{3}\right)\varphi'\left(\frac{1}{3}\right)\right]} \times \frac{1}{y_1 y_2 y_3}. \end{aligned}$$

Le premier facteur étant une constante pour des fonctions f et φ données, on a le théorème suivant :

Quand f et φ désignent deux fonctions quelconques données, le centre de courbure d'une courbe (8) au centre de gravité du triangle divise dans un rapport constant le rayon de courbure de la conique circonscrite qui la touche en ce point.

Si $f(\mu)=\mu$ et $\varphi(\mu)=\mu^2$, on retrouve le théorème établi au paragraphe 1 ; plus généralement si $f(\mu)=\mu^n$ et $\varphi(\mu)=\mu^{2n}$ on trouve cette généralisation de la propriété des cubiques (1) démontrée plus haut :

Le centre de courbure de la courbe

$$\sum a_i \mu_i^n (\mu_i^{2n} - \mu_i^{2p}) = 0$$

au centre de gravité du triangle de référence est le même que celui de la courbe anharmonique qui la touche en ce point.

Enfin, des développements qui précèdent et du résultat du paragraphe 2 résulte encore le théorème suivant :

Quand f et φ désignent des fonctions quelconques données, le lieu géométrique des centres de courbure des courbes (8) correspondant au centre de gravité du triangle est une cubique, qui devient une trisectrice de de Longchamps quand le triangle est équilatéral.

On the Null-system,

by

YOSHITOMO OKADA, Sendai.

The object of this paper is to find some properties of certain geometrical figures gotten from the null-system concerning a given system of forces. First we prove the theorem that the reciprocal polars of the tangents to a circle with its center at the null-point of a plane generate an oblique circular cone whose vertex is at the null-point and one system of whose circular sections is parallel to the plane. If we call the circle, the cone and the locus of centers of this system of circular sections, the null-circle of the plane, the null-cone of the null-circle on the plane and the axis of the null-cone respectively, we find that the axis is independent of the radius of the null-circle and that only one null-cone on the plane is orthogonal in the sense of Schröter. Then we call the two generating lines perpendicular to the two systems of parallel circular sections of this orthogonal null-cone its principal generator and bigenerator, and study the relations among the axis and these two particular generators. These two generators characterize the orthogonal null-cone, so that it seems to be worth while to study the properties of these lines. Since the principal generator, however, is the same line as the "Verschiebungslinie" in Prof. H. E. Timerding's *Geometrie der Kräfte* and has been already studied in that work by him, we have tried to study the so-called bigenerator only.

1. Let the co-ordinates of a given system of forces referred to a rectangular co-ordinate system be $X, Y, Z; L, M, N$. Then⁽¹⁾ we have for the null-point (x_0, y_0, z_0) of the plane

$$(1) \quad uv + vy + wz = 1,$$
$$(2) \quad \begin{cases} x_0 = \frac{X + Nr - Mw}{Xu + Yv + Zw}, \\ y_0 = \frac{Y + Lw - Nu}{Xu + Yv + Zw}, \end{cases}$$

(1) H. E. Timerding, *Geometrie der Kräfte*, 1908, pp. 92 and 93. I greatly owe to this excellent work during my investigation, and refer to it several times throughout this paper by the abbreviation Tim., G. d. K.

$$\left\{ \begin{array}{l} z_0 = \frac{Z + Mu - Lv}{Xu + Yv + Zw}; \end{array} \right.$$

while if equation (1) be taken as the null-plane of point (x_0, y_0, z_0) ,

$$(3) \quad \left\{ \begin{array}{l} u = \frac{L - Zy_0 + Yz_0}{Lx_0 + My_0 + Nz_0}, \\ v = \frac{M - Xz_0 + X_0z_0}{Lx_0 + My_0 + Nz_0}, \\ w = \frac{N - Yx_0 + Xy_0}{Lx_0 + My_0 + Nz_0}. \end{array} \right.$$

From these equations, it can be easily proved that all the null-planes of points on a straight line p_1 pass through a fixed straight line p_2 and vice versâ; so that we call p_1 and p_2 *reciprocal polars* as usual.

Now, take any plane π in space and take a rectangular co-ordinates system with it as xy -plane. Then for the given system of forces whose co-ordinates are $X, Y, Z; L, M, N$, the co-ordinates of the null-point of plane π become

$$(4) \quad x_0 = -M/Z, \quad y_0 = L/Z, \quad z_0 = 0.$$

Hence if a circle with center at the null-point of a plane and with any radius be called a *null-circle* of the plane, the equations to a null-circle (radius r) of plane π are in parametric form

$$(5) \quad x = -M/Z + r \cos \theta, \quad y = L/Z + r \sin \theta, \quad z = 0,$$

and the equation to the surface generated by the reciprocal polars of the tangents to the null-circle (5) is

$$(6) \quad (Zy - Yz - L)^2 + (Xz - Zv - M)^2 = \frac{T^2}{Z^2 r^2} z^2,$$

where

$$T = XL + YM + ZN.$$

For, the equations to any tangent to the circle (5) are

$$\begin{aligned} x_0 &= -M/Z + r \cos \theta - t \sin \theta, \\ y_0 &= L/Z + r \sin \theta + t \cos \theta, \\ z_0 &= 0, \end{aligned}$$

t being a parameter, and x_0, y_0 and z_0 being the current co-ordinates of a point. Substituting these values of x_0, y_0, z_0 in the equation to the null-plane of point (x_0, y_0, z_0) gotten from (1) and (3), we have

$$r Zx \sin \theta - r Zy \cos \theta - \left(\frac{T}{Z} - r Y \cos \theta + r X \sin \theta \right) z + r L \cos \theta + r M \sin \theta \\ + (Zx \cos \theta + Zy \sin \theta - (Y \sin \theta + X \cos \theta) z - L \sin \theta + M \cos \theta) t = 0.$$

Hence the reciprocal polar of the tangent has the equations

$$\begin{cases} r Zx \sin \theta - r Zy \cos \theta - (T/Z - r Y \cos \theta + r X \sin \theta) z + r L \cos \theta + r M \sin \theta = 0, \\ Zx \cos \theta + Zy \sin \theta - (Y \sin \theta + X \cos \theta) z - L \sin \theta + M \cos \theta = 0, \end{cases}$$

i. e.

$$\begin{cases} (Xz - Zx - M) \sin \theta + (Yy - Yz - L) \cos \theta = -\frac{T}{Zr} z, \\ (Zy - Yz - L) \sin \theta - (Xz - Zx - M) \cos \theta = 0. \end{cases}$$

Eliminating θ from these equations we arrive at equation (6).

It is easily seen that the surface (6) is an oblique circular cone having its vertex at the null-point (4) and that the equations of the two systems of its parallel circular sections are

$$(7) \quad z = a,$$

$$(8) \quad 2ZXx + 2YZy + \left(\frac{T^2}{Z^2 r^2} - X^2 - Y^2 + Z^2 \right) z + 2MX - 2LY = \beta,$$

where a and β are two arbitrary constants.

If we call the cone (6) the *null-cone* of the null-circle (5) on plane π , we have the following result:—*The null-cone of a null-circle on any plane is an oblique circular cone and one system of its circular sections is parallel to the plane, the null-point of the plane, the center of the null-circle and the vertex of the null-cone being coincident.*

2. We will now investigate the null-cones of null-circles on plane π with special radii.

1. If radius r become infinite, then the corresponding null-circle becomes the line at infinity on plane π ; and consequently its null-cone is the reciprocal polar of that line at infinity and has the equations

$$(9) \quad \frac{x + M/Z}{X} = \frac{y - L/Z}{Y} = \frac{z}{Z}.$$

On the other hand, we know that the locus of centers of the system of circular sections parallel to plane π is the straight line (9). If we call this locus the *axis* of the null-cone (6), then, since (9) is independent of r , we can say that all the null-cones of the null-circles on a plane have a common axis, that is the reciprocal polar of the line at infinity on the plane, passes through the null-point of the plane and has the direction

cosines (referred to the rectangular co-ordinate system, x - and y -axes lying in the plane and z -axis perpendicular to it) proportional to the components of the given system of forces along the axes of co-ordinates.

2. If radius r be 0, then the null-circle becomes the null-point of the plane π and its null-cone becomes the plane π . Hence if we consider a point as a circle with radius zero, then the null-cone of the null-point of a plane is that plane itself.

3. If radius $r = T/(\sqrt{X^2 + Y^2})$ numerically, then the null-cone has the equation

$$(10) \quad (Zy - Yz - L)^2 + (Xz - Zx - M)^2 = (X^2 + Y^2) z^2,$$

of which two straight lines

$$(11) \quad \frac{x + M/Z}{0} = \frac{y - L/Z}{0} = \frac{z}{1},$$

$$(12) \quad \frac{x + M/Z}{2X} = \frac{y - L/Z}{2Y} = \frac{z}{Z},$$

are two generating lines, and from (7) and (8) the two systems of its parallel circular sections have the equations

$$(13) \quad z = \alpha,$$

$$(14) \quad 2Xx + 2Yy + Zz = \beta,$$

α and β being two arbitrary constants. Therefore the straight lines (11) and (12) are perpendicular to the planes (13) and (14) respectively. Hence the null-cone (10) is *orthogonal* ⁽¹⁾ whose two generating lines are perpendicular to the two systems of its parallel circular sections (13) and (14). The generating line (11), which is perpendicular to the plane π at the null-point, is the reciprocal polar of a certain tangent to the null-circle with null-cone (10). Conversely a certain tangent to the null-circle with null-cone (10) is the reciprocal polar of the generating line (11). Hence if the reciprocal polar of the straight line, perpendicular to any plane at its null-point, be a tangent to a null-circle on the plane, then the null-cone of that null-circle is orthogonal.

We call the generating lines (11) and (12) the *principal generator* and *bigenerator* of the orthogonal null-cone (10) respectively. Then from (9), (11) and (12) we can prove that the axis, principal generator and bigenerator of the orthogonal null-cone (10) lie in the plane

⁽¹⁾ According to the denomination of Schröter, Journ. f. Math., Bd. 85, 1878, pp. 41 and 79. Also see Tim., G. d. K., p. 114.

$$(15) \quad Y\left(x + \frac{M}{Z}\right) = X\left(y - \frac{L}{Z}\right).$$

Let the angles between two of the three straight lines (9), (11), (12) be φ_{a1} , φ_{12} , φ_{2a} , then

$$(16) \quad \begin{cases} \cos^2 \varphi_{a1} = \frac{Z^2}{X^2 + Y^2 + Z^2}, \\ \cos^2 \varphi_{12} = \frac{Z^2}{4(X^2 + Y^2) + Z^2}, \\ \cos^2 \varphi_{2a} = \frac{\{2(X^2 + Y^2) + Z^2\}^2}{(X^2 + Y^2 + Z^2)\{4(X^2 + Y^2) + Z^2\}}. \end{cases}$$

3. We will next obtain the relations among the moments of the principal generator (11), the bigenerator (12), the axis of the orthogonal null-cone (10) and the null-circle⁽¹⁾ with null-cone (10), with respect to the given system of forces $(X, Y, Z; L, M, N)$. If we denote these moments by \mathfrak{M}_1 , \mathfrak{M}_2 , \mathfrak{M}_a and θ_0 respectively, then, since the line (11) is perpendicular to plane π at the null-point,

$$(17) \quad \mathfrak{M}_1^2 = (L - Zy_0 + Yz_0)^2 + (M - Xz_0 + Zx_0)^2 + (N - Yx_0 + Xy_0)^2,$$

where x_0 , y_0 and z_0 are the co-ordinates of the null-point of plane π . Hence by (4)

$$(18) \quad \mathfrak{M}_1^2 = T^2/Z^2.$$

The moment \mathfrak{M} of a straight line making an angle φ with the line (11) and passing through the null-point of plane π with respect to the given system of forces is represented by the equation

$$(19) \quad \mathfrak{M} = \mathfrak{M}_1 \cos \varphi,$$

so that

$$\mathfrak{M}_2^2 = \mathfrak{M}_1^2 \cos^2 \varphi_{12},$$

whence

$$(20) \quad \mathfrak{M}_2^2 = T^2/\{4(X^2 + Y^2) + Z^2\}.$$

Similarly

$$(21) \quad \mathfrak{M}_a^2 = T^2/(X^2 + Y^2 + Z^2).$$

The moment θ of a null-circle (radius r) on the plane π with respect to the given system of forces is represented by the equation

(1) The moment of null-circle means the moment of tangent to null-circle.

(2) By equation (19) we know that the moment of the straight line which is perpendicular to a plane at the null-point is the greatest of all the moments of the other straight lines passing through that same point.

$$(22) \quad \theta = Zr^{(1)},$$

so that

$$(23) \quad \theta_0^2 = T^2 / (X^2 + Y^2).$$

Therefore from (18), (21) and (23), we have

$$(24) \quad \frac{1}{\theta_0^2} + \frac{1}{\mathfrak{M}_1^2} = \frac{1}{\mathfrak{M}_a^2},$$

and from (18), (20) and (21),

$$(25) \quad \frac{1}{\mathfrak{M}_2^2} + \frac{3}{\mathfrak{M}_1^2} = \frac{1}{\mathfrak{M}_a^2}.$$

These two equations are the required relations among the named four moments.

4. Thus far we have concerned with a fixed plane in space, but we will consider with interest the case where the plane is moving according to some law.

For convenience, we take the central axis of the system of forces as z -axis of the co-ordinates system, so that $X=Y=L=M=0$. By the central axis of the system of forces we understand as follows⁽²⁾. From (17) the moment \mathfrak{M} at a point (x, y, z) (which means the greatest of the moments of all the straight lines passing through the point (x, y, z)) is represented by the equation

$$\mathfrak{M}^2 = (L - Zy + Yz)^2 + (M - Xz + Zx)^2 + (N - Yx + Xy)^2.$$

Differentiating with respect to x, y and z respectively, and putting all the results equal to zero

$$(M - Xz + Zx) Z = (N - Yx + Xy) Y,$$

$$(N - Yx + Xy) X = (L - Zy + Yz) Z,$$

$$(L - Zy + Yz) Y = (M - Xz + Zx) X.$$

Since these equations are not independent they represent a straight line. From the construction of the equations the moment of any point on that line is extremum. This straight line shall be called the *central axis* of the system of forces.

If we take this central axis as z -axis, $x=y=0$, so that

$$X=Y=L=M=0,$$

and consequently the resultant force of our system of forces is

(1) Tim., G. d. K., p. 94.

(2) Tim., G. d. K., p. 100.

$$R = \sqrt{X^2 + Y^2 + Z^2} = Z,$$

and its sense is the same as the positive sense of z -axis, so that R is always positive.

Referred to this new co-ordinate system, the equations (2) for the null-point (x_0, y_0, z_0) of the plane

$$(1) \quad ux + vy + wz = 1$$

become

$$(26) \quad x_0 = k v/w, \quad y_0 = -k u/w, \quad z_0 = 1/w;$$

while if the equation of the null-plane of a point (x_0, y_0, z_0) be taken in the form (1), then

$$(27) \quad u = -y_0/kz_0, \quad v = x_0/kz_0, \quad w = 1/z_0;$$

where $k = N/Z = Z/R$. This constant k is the parameter of the null-system.

Now, taking any plane in the form (1), and passing through any point on the plane, draw three straight lines whose direction-cosines are proportional to l_1, l_2, l_3 ; m_1, m_2, m_3 ; and n_1, n_2, n_3 , where

$$(28) \quad \begin{cases} l_1 = 0, & l_2 = w, & l_3 = -v; \\ m_1 = -(v^2 + w^2), & m_2 = u v, & m_3 = u w; \\ n_1 = u, & n_2 = v, & n_3 = w. \end{cases}$$

Then

$$\sum_{i=1}^3 m_i n_i = 0, \quad \sum_{i=1}^3 n_i l_i = 0, \quad \sum_{i=1}^3 l_i m_i = 0.$$

Therefore these three lines are perpendicular to each other. If we denote the components of the system of forces along these three lines by X', Y' and Z' ,

$$(29) \quad X' = \frac{-vR}{(v^2 + w^2)^{\frac{1}{2}}}, \quad Y' = \frac{uwR}{\{(v^2 + w^2)(u^2 + v^2 + w^2)\}^{\frac{1}{2}}}, \\ Z' = \frac{wR}{(u^2 + v^2 + w^2)^{\frac{1}{2}}}.$$

Take these three lines as x' -, y' - and z' -axes. Then, from (9), (11) and (12), the direction-cosines, referred to these axes, of the axis and the principal generator and the bigenerator of the orthogonal null-cone of the plane (1) become proportional to X', Y', Z' ; $0, 0, 1$; and $2X', 2Y', Z'$. But, from (28), referred to x' -, y' - and z' -axes, the direction-cosines of x -, y - and z -axes are

$$\begin{aligned}
& 0, \quad -\frac{v^2 + w^2}{\{(v^2 + w^2)(u^2 + v^2 + w^2)\}^{\frac{1}{2}}}, \quad \frac{u}{(u^2 + v^2 + w^2)^{\frac{1}{2}}}; \\
& \frac{w}{(v^2 + w^2)^{\frac{1}{2}}}, \quad \frac{uv}{\{(v^2 + w^2)(u^2 + v^2 + w^2)\}^{\frac{1}{2}}}, \quad \frac{v}{(u^2 + v^2 + w^2)^{\frac{1}{2}}}; \\
& \frac{-v}{(v^2 + w^2)^{\frac{1}{2}}}, \quad \frac{uw}{\{(v^2 + w^2)(u^2 + v^2 + w^2)\}^{\frac{1}{2}}}, \quad \frac{w}{(u^2 + v^2 + w^2)^{\frac{1}{2}}};
\end{aligned}$$

so that the axis and the two generators have the following direction-cosines referred to x -, y - and z -axes:

$$\begin{aligned}
& 0, \quad 0, \quad 1; \\
& \frac{u}{(u^2 + v^2 + w^2)^{\frac{1}{2}}}, \quad \frac{v}{(u^2 + v^2 + w^2)^{\frac{1}{2}}}, \quad \frac{w}{(u^2 + v^2 + w^2)^{\frac{1}{2}}}; \\
& -uw/\sqrt{S}, \quad -vw/\sqrt{S}, \quad \{2(u^2 + v^2) + w^2\}/\sqrt{S};
\end{aligned}$$

where

$$S = u^2 w^2 + v^2 w^2 + \{2(u^2 + v^2) + w^2\}^2.$$

Since these three lines pass through the null-point (26) of the plane (1), their equations become

$$\begin{aligned}
\frac{x - kv/w}{0} &= \frac{y + ku/w}{0} = \frac{z - 1/w}{1}, \\
\frac{x - kv/w}{u} &= \frac{y + ku/w}{v} = \frac{z - 1/w}{w}, \\
\frac{x - kv/w}{-uv} &= \frac{y + ku/w}{-vw} = \frac{z - 1/w}{2(u^2 + v^2) + w^2};
\end{aligned}$$

or when we express these equations in terms of the co-ordinates of the null-point (x_0, y_0, z_0) , by using (26) and (27), we have

$$(30) \quad \left\{ \begin{aligned} \frac{x - x_0}{0} &= \frac{y - y_0}{0} = \frac{z - z_0}{1}, \\ \frac{x - x_0}{-y_0} &= \frac{y - y_0}{x_0} = \frac{z - z_0}{k}, \\ \frac{x - x_0}{y_0 k} &= \frac{y - y_0}{-x_0 k} = \frac{z - z_0}{2(x_0^2 + y_0^2) + k}. \end{aligned} \right.$$

5. We will rotate the plane considered around a fixed axis. In this case, the locus of the null-point (x_0, y_0, z_0) of that plane is the re-

iprocal polar of that axis. Hence, let the equation to the reciprocal polar of that axis be

$$(31) \quad x_0 = lz_0 + a, \quad y_0 = mz_0 + b.$$

From (30), the equation to the plane, which contains the axis and the two generators of the orthogonal null-cone of the null-plane of a point $(x_0, y_0, z_0)^{(1)}$, is

$$(32) \quad x_0(x - x_0) + y_0(y - y_0) = 0.$$

Therefore, from (31) and (32) we know that when the plane rotates around a fixed axis (its reciprocal polar being (31)) the plane envelops the surface

$$(33) \quad \{2(la + mb) - (lx + my)\}^2 + 4(l^2 + m^2)(ax + by - a^2 - b^2) = 0,$$

where l, m, a, b can not be zero at the same time.

This surface is a cylinder of the second order. The condition that it may be broken up into two planes is

$$(34) \quad (l^2 + m^2)(am - bl) = 0,$$

i. e.

$$1. \quad l = m = 0,$$

$$2. \quad a = b = 0,$$

or,

$$3. \quad \frac{l}{a} = \frac{m}{b} = p (= \text{const.}).$$

When any of these conditions is satisfied we can verify that equation (33) expresses two coincident planes.

Case 1. In this case, (31) become

$$x = a, \quad y = b,$$

and consequently the axis of rotation is the line at infinity

$$ay - bx + kz = 0, \quad k = 0 \quad (2).$$

(1) x_0, y_0 are not zero at the same time, for if so the plane (32) becomes indeterminate.

(2) From (1) and (27), the null-plane of a point (x_0, y_0, z_0) is

$$(i) \quad y_0 x - x_0 y - k(z - z_0) = 0,$$

as is well known. The reciprocal polar of (31) (i. e. the axis of rotation) is obtained from equation (i) and

$$(ii) \quad x_0 = lz_0 + a, \quad y_0 = mz_0 + b,$$

substituting (ii) into (i),

$$bx - ay - kz + (mx - ly + k)z_0 = 0.$$

Therefore the axis of rotation is

$$bx - ay - kz = 0, \quad mx - ly + k = 0.$$

Hence the rotation in this case is the translation of the plane.

Case 2. Equations (31) become

$$x=lz, \quad y=mz,$$

and consequently the axis of rotation is

$$mx - ly + k = 0, \quad z = 0;$$

so that this axis of rotation is perpendicular to the central axis.

Case 3. Equations (31) become

$$x=apz+a, \quad y=bpz+b,$$

and consequently the axis of rotation is

$$\begin{cases} bx - ay - kz = 0, \\ pbx - pay + k = 0; \end{cases}$$

i. e.

$$bx - ay + k/p = 0, \quad z + 1/p = 0;$$

so that this axis of rotation is also perpendicular to the central axis.

But, of these three cases, the case in which the axis of rotation is the line at infinity on a system of parallel planes normal to the central axis does not occur, since all of l , m , a , b can not be zero at the same time. Hence if the surface (33) degenerates into two coincident planes, the axis of rotation must be the line at infinity (not lying on planes normal to the central axis) or perpendicular to the central axis. The converse of this theorem is also true, and moreover, we know that the surface (33) is a parabolic cylinder whose generating lines are parallel to the central axis, excepting the case in which (33) becomes two coincident planes. Therefore the plane, on which the axis and the two generators of the orthogonal null-cone* on each plane of an affine axial pencil, or an axial pencil whose axis is not a line at infinity and is perpendicular to the central axis of the given system of forces, lie, in a fixed plane parallel to the central axis; and that of any other axial pencil, except an axial pencil whose axis is the line at infinity on the planes perpendicular to the central axis, envelops a parabolic cylinder whose generating lines are parallel to the central axis of the given system of forces.

6. As we have already seen, on the null-plane of a point, there is an orthogonal null-cone whose principal generator and bigenerator pass through that point. We will find a certain properties of the bigenerator,

while the principal generator has been already studied in Prof. H. E. Timmerding's *Geometrie der Kräfte*⁽¹⁾.

The point at which the bigenerator meets the null-plane we will call its *foot*. Then from (30) the equations to the bigenerator at the foot (x_0, y_0, z_0) are

$$(35) \quad \frac{x-x_0}{y_0 k} = \frac{y-y_0}{-x_0 k} = \frac{z-z_0}{2(x_0^2+y_0^2)+k} (= \varphi \text{ say}),$$

which may be written in the form

$$(36) \quad x=rz+\rho, \quad y=sz+\sigma,$$

if we put

$$(37) \quad \begin{cases} r = \frac{y_0 k}{2(x_0^2+y_0^2)+k^2}, & \rho = x_0 - \frac{y_0 z_0 k}{2(x_0^2+y_0^2)+k^2}, \\ s = \frac{-x_0 k}{2(x_0^2+y_0^2)+k^2}, & \sigma = y_0 + \frac{z_0 x_0 k}{2(x_0^2+y_0^2)+k^2}. \end{cases}$$

From (37),

$$(38) \quad x_0 = \frac{s\eta}{s^2+r^2}, \quad y_0 = \frac{-r\eta}{s^2+r^2}, \quad z_0 = -\frac{s\sigma+\rho r}{s^2+r^2};$$

where

$$\eta = \rho s - r \sigma.$$

We can verify that (37) satisfy

$$(39) \quad 2\eta^2 + k\eta + k^2(s^2+r^2) = 0.$$

Hence the bigenerator at the foot (x_0, y_0, z_0) belongs to the quadratic line-complex (39). Conversely any line belonging to the complex (39) is a bigenerator at the foot (x_0, y_0, z_0) . Equation (39) being a quadratic equation with respect to k , a bigenerator belongs in general to two quadratic complexes corresponding to the two values of k . That is to say, a bigenerator of one null-system is in general also a bigenerator of another null-system with the same central axis. But in the case $k=k'$, a bigenerator belongs to only one null-system. Such a line belongs to a line-congruence

$$\begin{cases} s^2+r^2=1/8, \\ 2\eta^2+k\eta+k^2(s^2+r^2)=0, \end{cases}$$

i. e.

(1) The principal generator of the orthogonal null-cone on the null-plane of a point (x_0, y_0, z_0) is perpendicular to that plane at that point, so that this line is the "Verschiebungslinie" in Tim. E. d. K. p. 103.

$$(40) \quad \begin{cases} s^2 + r^2 = 1/8, \\ 4\eta + k = 0. \end{cases}$$

From (38) and (40) we know that the locus of the foot (x_0, y_0, z_0) of such a line is a right circular cylinder with the central axis as its axis and with the equation

$$(41) \quad x_0^2 + y_0^2 = \frac{k^2}{2}.$$

Therefore the foot of a bigenerator belonging to only one null-system lies on a fixed right circular cylinder with the central axis of the given system of forces as its axis.

It is well known that all the lines of a quadratic line-complex passing through a fixed point form a cone of the second degree with the point as its vertex. In our case, all the lines of the complex (39) passing through a fixed point (x, y, z) form a cone of the second degree

$$(42) \quad 2(y'x - x'y)^2 + k(y'x - x'y)(z' - z) + k^2(x' - x)^2 + k^2(y' - y)^2 = 0,$$

where x' , y' and z' are current co-ordinates. For, we obtain (42) by substituting

$$\eta = \frac{y'x - x'y}{z' - z}, \quad r = \frac{x' - x}{z' - z}, \quad s = \frac{y' - y}{z' - z}$$

into (39). On the other hand, if we put φ instead of $k\varphi$ in (35),

$$(43) \quad \begin{aligned} x_0 &= (x - \varphi y)/(1 + \varphi^2), \\ y_0 &= (y + \varphi x)/(1 + \varphi^2), \\ z_0 &= \frac{kz - (2x^2 + 2y^2 + k^2)\varphi + k\varphi^2 - k^2\varphi^3}{k(1 + \varphi^2)}; \end{aligned}$$

where x , y and z are constant and φ is a parameter. These express the locus of the feet of the bigenerator of the null-system passing through the fixed point (x, y, z) . Hence (43) must satisfy (42). From the first and second equations of (43), we have

$$(44) \quad x_0^2 + y_0^2 - xx_0 - yy_0 = 0,$$

which represents a right circular cylinder, so that the curve is either the complete intersection or partial intersection of this cylinder (44) and the cone (42). The complete intersection is a space quartic, which consists of a straight line and a space cubic, but that straight line can be no part of the required curve. Hence the curve (43) is a space cubic on the cone (42).

From (44), we have

$$(45)_1 \quad \begin{cases} x_0 = x/2 + r \cos \theta, \\ y_0 = y/2 + r \sin \theta, \end{cases}$$

where $r = \sqrt{x^2 + y^2}/2$ and θ is a parameter; and from $(45)_1$ and (43) , we have

$$(45)_2 \quad z_0 = z - k \sin(\theta - \tau)/(1 + \cos(\theta - \tau)) - \frac{4}{k} r^2 \sin^2(\theta - \tau),$$

where

$$x = 2r \cos \tau, \quad y = 2r \sin \tau.$$

When we develop the cylinder (44) on a plane, the space cubic curve on it becomes a plane curve, which can be traced out by using $(45)_1$ and $(45)_2$ for a given set of values of k and r . The cubic passes through the given fixed point (x, y, z) .

We know using (40) that there exist two bigenerators belonging to only one null-system passing through the given point (x, y, z) . Therefore, among the bigenerators of a given null-system passing through a fixed point there are only two belonging to that null-system only and a cone of the second degree is formed by them; and their feet form a space cubic curve on that cone.

7. It is well known that the lines of a quadratic line-complex lying on a given plane envelop a curve of the second class. In our case, the feet of the tangents to the curve enveloped by the lines of the complex (39) lying on a given plane, i. e. the feet of the bigenerators of the given null-system lying on a given plane, form a curve on the plane. At first let us find out this curve.

By a rotation of the x - and y -axes around the z -axis and by a translation of the xy -plane in the direction of the z -axis, we can bring the equation of any given plane in the form

$$(46) \quad z = ay.$$

From (35) a bigenerator of the given null-system at the foot (x_0, y_0, z_0) is

$$(35)' \quad \begin{cases} x = x_0 + y_0 k \varphi, \\ y = y_0 - x_0 k \varphi, \\ z = z_0 + 2(x_0^2 + y_0^2) \varphi + k^2 \varphi. \end{cases}$$

When such a line lies on the plane (46), the relation

$$\begin{aligned} z_0 + 2(x_0^2 + y_0^2) \varphi + k^2 \varphi &= a(y_0 - x_0 k \varphi), \\ z_0 - a y_0 + (2x_0^2 + 2y_0^2 + a k x_0 + k^2) \varphi &= 0 \end{aligned}$$

must be satisfied by every value of φ . Hence

$$z_0 - ay_0 = 0,$$

$$2(x_0^2 + y_0^2) + akx_0 + k^2 = 0,$$

i. e.

$$(47) \quad \begin{cases} z_0 = ay_0, \\ \left(x_0 + \frac{a}{4}k\right)^2 + y_0^2 = \frac{a^2 - 8}{16} k^2. \end{cases}$$

Equations (47) express the locus of the feet of the bigenerators of the given null-system lying on the given plane (46). This locus is in general an ellipse with x -axis as one of its axes. Therefore the feet of the bigenerators of the given null-system lying on a given plane form in general an ellipse with a straight line perpendicular to the central axis of the given system of forces as one of its axes.

Next, we will find out the bigenerators of the given null-system lying on the plane (46), and the curve enveloped by these lines, i. e. the curve of the second class enveloped by the lines of the complex (30) lying on the given plane (46).

As a bigenerator of the given null-system lying on (46) also lies by (35) on the plane

$$\frac{x - x_0}{y_0} = \frac{y - y_0}{-x_0},$$

the equations to that line are

$$(48) \quad \frac{x - x_0}{y_0} + \frac{y - y_0}{x_0} = 0, \quad z = ay,$$

where x_0 and y_0 are the co-ordinates of a point on the curve (47). We transform the co-ordinates system (x, y, z) into the new one (x, y', z') , such that the given plane be the xy' -plane of the new system. Then since the transformation of co-ordinates is expressible by

$$\begin{aligned} x &= x', \\ y &= y' \cos a' - z' \sin a', \\ z &= y' \sin a' + z' \cos a', \\ a &= \tan a', \end{aligned}$$

equation (47) becomes

(1) If $a^2 - 8 \leq 0$, the curve represented by (47) is imaginary or one point. Hence we consider only the case where $a^2 - 8$ is positive.

$$(47)' \quad \begin{cases} z_0' = 0, \\ \left(x_0 + \frac{ak}{4}\right)^2 + \frac{y_0'^2}{1+a^2} = \frac{a^2-8}{16} k^2; \end{cases}$$

and (48) becomes

$$(48)' \quad (x-x_0)x_0 + y_0'(y'-y_0')\cos^2 a' = 0, \quad z' = 0.$$

Hence we obtain as the envelope of the lines (48)'

$$\frac{16(x+ak/4)^2}{(a^2-8)k^2} - \frac{2y'^2}{(1+a^2)k^2} = 1,$$

that is in general a hyperbola whose asymptotes have the angular coefficients $\pm m$ which are determined by

$$m^2 = 8(1+a^2)/(a^2-8);$$

so that the angle between the two asymptotes is independent of the parameter of the null-system. From (41) and the second equation of (47) we get the abscissa $x_0 = -2k/a$ of the foot of a bigenerator belonging to only one given null-system on the given plane; hence by (47)' the coordinates of these points are

$$x_0 = -2k/a, \quad y_0' = \pm \sqrt{a^2-8} \sec a'k/(\sqrt{2}a), \quad z_0' = 0.$$

Therefore the number of these points is two, and consequently by (48)' the angle 2θ between the two bigenerators of the given null-system passing through these points respectively is expressed by

$$\tan^2 \theta = x_0^2/(y_0'^2 \cos^2 a') = 8/(a^2-8).$$

This angle is also independent of the parameter of the null-system. Therefore the bigenerators of the given null-system on a given plane envelop a hyperbola in general, whose asymptotes make an angle independent of the parameter of the null-system and one of whose axes is perpendicular to the central axis of the given system of forces; and only two of the bigenerators of the given null-system belong to that null-system only and make an angle independent of the parameter of the null-system.

As the bigenerators of the given null-system belonging to a line complex, those lines intersecting a fixed straight line belong to a line-congruence, so that there are ∞^2 of such lines; and since there exists one foot corresponding to such a bigenerator, all the feet of these bigenerators intersecting a fixed straight line form a surface. The intersection of that surface with a plane through the fixed straight line consists of that line and the locus of all the feet of bigenerators of the given null-system on that plane.

Hence if a plane through a fixed straight line be $z=ay$, what we can assume without loss of generality, then the intersection of the surface with this plane must be the fixed line and the ellipse (47) in general. Therefore we know that, if that plane be fixed, a part of the above intersection, the ellipse, is invariant, however the line be assumed on that plane.

8. All the bigenerators of the given null-system whose feet lie on a fixed straight line form a ruled surface. We will study this surface.

By a certain rotation of the x - and y -axes around the z -axis and by a sliding of the xy -plane along the z -axis, we can write the equation of a fixed straight line in the form

$$\frac{x_0 - a}{0} = \frac{y_0}{m} = \frac{z_0}{1} = t,$$

or

$$x_0 = a, \quad y_0 = mt, \quad z_0 = t.$$

From these equations and (35)', we obtain the following equations to that ruled surface:

$$(49) \quad \begin{cases} x = a + mkt\varphi, \\ y = mt - ak\varphi, \\ z = t + 2(a^2 + m^2 t^2)\varphi + k^2\varphi. \end{cases}$$

This surface is clearly a cubic; and one system of parametric lines $\varphi = \text{const.}$ are parabolas, except the given fixed line, and the system of lines $t\varphi = \text{const.}$ on it are hyperbolas. If we will determine the locus of the vertices of the parabolas, find the expression to the curvature of these parabolas,

$$\frac{1}{\rho^2} = 16m^6(1+k\varphi^2)\varphi^2/\{m^2(1+k^2\varphi^2) + (1+4m^2\varphi t)^2\}^3,$$

and differentiate this equation with respect to t and put the result equal to zero; then we have

$$\varphi t = -1/4m^2 = \text{const.},$$

which is a hyperbola.

Binary Forms and Duality,

by

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Clebsch⁽¹⁾ established the relation among 3 binary n -ic forms and the Jacobians of the Jacobians formed from the given forms; this result has been extended by Rosanes⁽²⁾ to the case of 4 binary n -ic forms and also by Lindemann⁽³⁾ to the case of 4 binary cubic forms. In this short note I will generalize these results to the case of r ($3 \leq r \leq n+1$) binary n -ic forms, *the method of proof being based entirely upon the principle of duality*, the starting point of Clebsch.

1. Take n binary n -ic forms:

$$f_1(\xi_1, \xi_2), \quad f_2(\xi_1, \xi_2), \quad \dots, \quad f_n(\xi_1, \xi_2)$$

and form from them the following determinants:

$$K(f_{i_1}, f_{i_2}, \dots, f_{i_m}) \equiv K_{i_1, i_2, \dots, i_m} \quad (i_1, i_2, \dots, i_m = 1, 2, \dots, n; 2 \leq m \leq n)$$

$$\equiv \begin{vmatrix} \frac{\partial^{m-1} f_{i_1}}{\partial \xi_1^{m-1}} & \frac{\partial^{m-1} f_{i_1}}{\partial \xi_1^{m-2} \partial \xi_2} & \dots & \frac{\partial^{m-1} f_{i_1}}{\partial \xi_2^{m-1}} \\ \frac{\partial^{m-1} f_{i_2}}{\partial \xi_1^{m-1}} & \frac{\partial^{m-1} f_{i_2}}{\partial \xi_1^{m-2} \partial \xi_2} & \dots & \frac{\partial^{m-1} f_{i_2}}{\partial \xi_2^{m-1}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial^{m-1} f_{i_m}}{\partial \xi_1^{m-1}} & \frac{\partial^{m-1} f_{i_m}}{\partial \xi_1^{m-2} \partial \xi_2} & \dots & \frac{\partial^{m-1} f_{i_m}}{\partial \xi_2^{m-1}} \end{vmatrix}.$$

(1) Clebsch, Über eine Eigenschaft von Funktionaldeterminanten, Journ. f. Math., 69 (1868), p. 355. See also Pascal, Die Determinanten (1900), p. 236.

(2) Rosanes, Über Funktionen, welche ein den Funktionaldeterminanten analoges Verhalten zeigen, Journ. f. Math., 75 (1873), p. 166.

(3) Lindemann, Über die Darstellung binärer Formen und ihrer Covarianten durch geometrische Gebilde im Raume, Math. Ann., 23 (1884), p. 111. He stated "... die in... aufgestellten entsprechenden Relationen für ein System von... cubischen Formen... auf Systeme binärer Formen n ter Ordnung erweitern lassen. Es würde mehr umständlich als schwierig sein,.... und es mag dies deshalb unterbleiben."

K_{i_1, i_2} is the Jacobian $J(f_{i_1}, f_{i_2})$ of f_{i_1} and f_{i_2} ; and K_{i_1, i_2, \dots, i_m} is the covariant which was called the Rosanesian by Prof. T. Hayashi⁽¹⁾. For the sake of brevity, when

$$f_{i_1} = K(f_{i_{11}}, f_{i_{12}}, \dots, f_{i_{1r}}), \dots, f_{i_m} = K(f_{i_{m1}}, f_{i_{m2}}, \dots, f_{i_{mr}}),$$

we will denote $K(f_{i_1}, f_{i_2}, \dots, f_{i_m})$ by

$$K_{i_{11} i_{12} \dots i_{1r}, \dots, i_{m1} i_{m2} \dots i_{mr}}.$$

It will be seen, by successive applications of Euler's formulae for homogeneous functions, that the following identities hold good:

$$\begin{aligned} \begin{vmatrix} f_{i_1} & df_{i_1} \\ f_{i_2} & df_{i_2} \end{vmatrix} &= \begin{vmatrix} f_{i_1} & \frac{\partial f_{i_1}}{\partial \xi_1} d\hat{\xi}_1 + \frac{\partial f_{i_1}}{\partial \xi_2} d\hat{\xi}_2 \\ f_{i_2} & \frac{\partial f_{i_2}}{\partial \xi_1} d\hat{\xi}_1 + \frac{\partial f_{i_2}}{\partial \xi_2} d\hat{\xi}_2 \end{vmatrix} = \frac{1}{n} K_{i_1, i_2} \begin{vmatrix} \hat{\xi}_1 & d\hat{\xi}_1 \\ \hat{\xi}_2 & d\hat{\xi}_2 \end{vmatrix} \\ &= \frac{1}{n} (\hat{\xi}_1 d\hat{\xi}_2 - \hat{\xi}_2 d\hat{\xi}_1) K_{i_1, i_2}, \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} f_{i_1} & df_{i_1} & d^2 f_{i_1} \\ f_{i_2} & df_{i_2} & d^2 f_{i_2} \\ f_{i_3} & df_{i_3} & d^2 f_{i_3} \end{vmatrix} &= \frac{1}{n(n-1)^2} K_{i_1, i_2, i_3} \begin{vmatrix} \hat{\xi}_1^2 & 2\hat{\xi}_1 d\hat{\xi}_1 & d\hat{\xi}_1^2 \\ \hat{\xi}_1 \hat{\xi}_2 & \hat{\xi}_2 d\hat{\xi}_1 + \hat{\xi}_1 d\hat{\xi}_2 & d\hat{\xi}_1 d\hat{\xi}_2 \\ \hat{\xi}_2^2 & 2\hat{\xi}_2 d\hat{\xi}_2 & d\hat{\xi}_2^2 \end{vmatrix}, \\ &= \frac{1}{n(n-1)} (\hat{\xi}_1 d\hat{\xi}_2 - \hat{\xi}_2 d\hat{\xi}_1)^3 K_{i_1, i_2, i_3}, \end{aligned}$$

.....

$$\begin{aligned} (1) \quad \begin{vmatrix} f_{i_1} & df_{i_1} & \dots & d^{m-1} f_{i_1} \\ f_{i_2} & df_{i_2} & \dots & d^{m-1} f_{i_2} \\ \dots & \dots & \dots & \dots \\ f_{i_m} & df_{i_m} & \dots & d^{m-1} f_{i_m} \end{vmatrix} &= \frac{\binom{m-1}{1} \binom{m-1}{2} \dots \binom{m-1}{m-2} K_{i_1, i_2, \dots, i_m}}{(m-1)^{m-2} n(n-1)^2 \dots (n-m+2)^{m-1}} \\ &\times \begin{vmatrix} \hat{\xi}_1^{m-1} & \binom{m-1}{1} \hat{\xi}_1^{m-2} d\hat{\xi}_1 & \dots & d\hat{\xi}_1^{m-1} \\ \hat{\xi}_1^{m-2} \hat{\xi}_2 & (m-2) \hat{\xi}_1^{m-3} \hat{\xi}_2 d\hat{\xi}_1 + \hat{\xi}_1^{m-2} d\hat{\xi}_2 & \dots & d\hat{\xi}_1^{m-2} d\hat{\xi}_2 \\ \hat{\xi}_1^{m-3} \hat{\xi}_2^2 & (m-3) \hat{\xi}_1^{m-4} \hat{\xi}_2^2 d\hat{\xi}_1 + 2\hat{\xi}_1^{m-3} \hat{\xi}_2 d\hat{\xi}_2 & \dots & d\hat{\xi}_1^{m-3} d\hat{\xi}_2^2 \\ \dots & \dots & \dots & \dots \\ \hat{\xi}_2^{m-1} & \binom{m-1}{1} \hat{\xi}_2^{m-2} d\hat{\xi}_2 & \dots & d\hat{\xi}_2^{m-1} \end{vmatrix} \\ &= \frac{\binom{m-1}{1} \binom{m-1}{2} \dots \binom{m-1}{m-2}}{(m-1)^{m-2} n(n-1)^2 \dots (n-m+2)^{m-1}} \cdot (\hat{\xi}_1 d\hat{\xi}_2 - \hat{\xi}_2 d\hat{\xi}_1)^{\frac{1}{2} m(m-1)} K_{i_1, i_2, \dots, i_m}. \end{aligned}$$

(1) Hayashi, Some theorems on binary forms, Science Reports of Tôhoku Imperial University, 6 (1917), p, 123.

2. Now since the case of 3 forms has been treated by Clebsch⁽¹⁾, we begin with the case of the 4 forms:

$$(2) \quad x_1 \equiv f_1(\xi_1, \xi_2), \quad x_2 \equiv f_2(\xi_1, \xi_2), \quad x_3 \equiv f_3(\xi_1, \xi_2), \quad x_4 \equiv f_4(\xi_1, \xi_2).$$

If we regard x_1, x_2, x_3, x_4 as the homogeneous point coordinates in space, (2) represents a rational curve of degree n , ξ_1, ξ_2 being homogeneous parameters. Then it follows from (1) that the plane coordinates of the osculating plane at the point (ξ_1, ξ_2) are given by

$$(3) \quad \begin{cases} \rho u_1 = K_{2,3,4} = K(x_2, x_3, x_4), \\ \rho u_2 = -K_{3,4,1} = -K(x_3, x_4, x_1), \\ \rho u_3 = K_{4,1,2} = K(x_4, x_1, x_2), \\ \rho u_4 = -K_{1,2,3} = -K(x_1, x_2, x_3). \end{cases}$$

Since (3) may be considered as the equations of the space curve (2) in the plane coordinates, the principle of duality shows us that

$$K(u_2, u_3, u_4), \quad -K(u_3, u_4, u_1), \quad K(u_4, u_1, u_2), \quad -K(u_1, u_2, u_3)$$

must be proportional to

$$x_1, \quad x_2, \quad x_3, \quad x_4$$

respectively. Hence

$$(4) \quad \frac{K_{341, 412, 123}}{f_1} = \frac{K_{412, 123, 234}}{f_2} = \frac{K_{123, 234, 341}}{f_3} = \frac{K_{234, 341, 412}}{f_4} = k^{(4)}.$$

Next we have from (1) the radial coordinates of the tangent to the space curve (2) at the point (ξ_1, ξ_2) :

$$\sigma p_{ik} = K_{i,k} = K(x_i, x_k), \quad (i, k = 1, 2, 3, 4);$$

and the axial coordinates of the tangent are

$$\begin{aligned} \sigma' q_{12} &= p_{43}, & \sigma' q_{23} &= p_{41}, & \sigma' q_{31} &= p_{42}, \\ \sigma' q_{41} &= p_{23}, & \sigma' q_{42} &= p_{31}, & \sigma' q_{43} &= p_{12}. \end{aligned}$$

(1) Clebsch's result is

$$\frac{K_{31,12}}{f_1} = \frac{K_{12,23}}{f_2} = \frac{K_{23,31}}{f_3} = k^{(3)}.$$

When f_1, f_2, f_3 are quadratic forms, the proportional factor $k^{(3)}$ becomes constant. If binary quadratic forms be referred to the *normal curve in a plane*, the above equations are equivalent to the following theorem, which is *self-evident*: In a plane, the polar triangle of the polar triangle of a given triangle, with respect to a fixed conic (the normal curve), coincides with the given triangle.

But by the principle of duality, q_{ik} must be proportional to $K(u_i, u_k)$; so that

$$(5) \quad \frac{K_{234, 341}}{K_{4, 3}} = \frac{K_{341, 412}}{K_{4, 1}} = \frac{K_{412, 234}}{-K_{4, 2}} \\ = \frac{K_{123, 234}}{K_{2, 3}} = \frac{K_{123, 341}}{-K_{3, 1}} = \frac{K_{123, 412}}{K_{1, 2}} = k_{II}^{(4)}.$$

When f_i ($i=1, 2, 3, 4$) are cubic forms, $k_I^{(4)}$ and $k_{II}^{(4)}$ become constant⁽¹⁾. If 4 binary cubic forms be referred to the *normal curve in space*⁽²⁾, (4) is equivalent to the following theorem, which is *self-evident*: The polar tetrahedron of a given tetrahedron, with respect to a fixed space cubic (the normal curve), coincides with the given tetrahedron. The given tetrahedron and its polar tetrahedron are mutually inscribed.

3. We pass now to consider the 5 forms:

$$(6) \quad x_1 \equiv f_1(\xi_1, \xi_2), \quad x_2 \equiv f_2(\xi_1, \xi_2), \quad x_3 \equiv f_3(\xi_1, \xi_2), \\ x_4 \equiv f_4(\xi_1, \xi_2), \quad x_5 \equiv f_5(\xi_1, \xi_2).$$

If we regard x_1, x_2, x_3, x_4, x_5 as the homogeneous point coordinates in space of 4 dimensions, (6) represents a rational curve of degree n , ξ_1, ξ_2 being homogeneous parameters. Then it follows from (1) that the hyperplane coordinates of the osculating hyperplane at the point (ξ_1, ξ_2) are given by

$$(7) \quad \begin{cases} \rho u_1 = K_{2, 3, 4, 5} = K(x_2, x_3, x_4, x_5), \\ \rho u_2 = K_{3, 4, 5, 1} = K(x_3, x_4, x_5, x_1), \\ \rho u_3 = K_{4, 5, 1, 2} = K(x_4, x_5, x_1, x_2), \\ \rho u_4 = K_{5, 1, 2, 3} = K(x_5, x_1, x_2, x_3), \\ \rho u_5 = K_{1, 2, 3, 4} = K(x_1, x_2, x_3, x_4). \end{cases}$$

But by the principle of duality,

$$K(u_2, u_3, u_4, u_5), \quad K(u_3, u_4, u_5, u_1), \quad K(u_4, u_5, u_1, u_2), \\ K(u_5, u_1, u_2, u_3), \quad K(u_1, u_2, u_3, u_4)$$

(1) For this case Lindemann gave

$$\frac{K_{123, 234, 341}}{f_3} = k_I^{(4)}, \quad \frac{K_{123, 234}}{K_{2, 3}} = k_{II}^{(4)};$$

and determined these two constants.

(2) Fr. Meyer, *Apolarität und rationale Curven* (1883), p. 46. If f_1, f_2, f_3 be given points in space, $K_{1, 2, 3}$ represents the pole of the plane passing through the given 3 points. For another interpretation of $K_{1, 2, 3}$, see Hayashi, loc. cit.

must be proportional to

$$x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5$$

respectively; so that

$$(8) \quad \frac{K_{3451, 4512, 5123, 1234}}{f_1} = \frac{K_{4512, 5123, 1234, 2345}}{f_2} = \frac{K_{5123, 1234, 2345, 3451}}{f_3} \\ = \frac{K_{1234, 2345, 3451, 4512}}{f_4} = \frac{K_{2345, 3451, 4512, 5123}}{f_5} = l_I^{(5)}.$$

Next we have from (1) the radial coordinates of the tangent to the curve at the point (ξ_1, ξ_2) :

$$\sigma p_{ik} = K_{i,k} = K(x_i, x_k), \quad (i, k=1, 2, 3, 4, 5).$$

Further let the plane

$$\begin{cases} v_1 x_1 + v_2 x_2 + v_3 x_3 + v_4 x_4 + v_5 x_5 = 0, \\ w_1 x_1 + w_2 x_2 + w_3 x_3 + w_4 x_4 + w_5 x_5 = 0 \end{cases}$$

have the contact of the second order to the curve (6) at the point (ξ_1, ξ_2) and let us put

$$\pi_{ik} = \begin{vmatrix} v_i & v_k \\ w_i & w_k \end{vmatrix}.$$

Then it follows from (1), by aid of Grassmann's theorem⁽¹⁾, that

$$\begin{aligned} \lambda \pi_{12} &= K_{3, 4, 5}, & \lambda \pi_{23} &= K_{1, 4, 5}, & \lambda \pi_{34} &= K_{1, 3, 5}, & \lambda \pi_{45} &= K_{1, 2, 3}, \\ \lambda \pi_{13} &= -K_{2, 4, 5}, & \lambda \pi_{24} &= -K_{1, 3, 5}, & \lambda \pi_{35} &= -K_{1, 2, 4}; \\ \lambda \pi_{14} &= K_{2, 3, 5}, & \lambda \pi_{25} &= K_{1, 3, 4}; \\ \lambda \pi_{15} &= -K_{2, 3, 4}. \end{aligned}$$

But by the principle of duality, $\pi_{ik}(x)$ must be proportional to $p_{ik}(u)$, and $p_{ik}(u)$ to $\pi_{ik}(x)$; consequently

$$(9) \quad \frac{K_{4512, 5123, 1234}}{K_{1, 2}} = \frac{K_{3451, 5123, 1234}}{-K_{1, 3}} = \frac{K_{3451, 4512, 1234}}{K_{1, 4}} \\ = \frac{K_{3451, 4512, 5123}}{-K_{1, 5}} = \frac{K_{2345, 5123, 1234}}{K_{2, 3}} = \dots = l_{II}^{(5)},$$

and

(¹) See Fr. Meyer, loc. cit., p. 1.

$$\begin{aligned}
 (10) \quad \frac{K_{2345, 3451}}{K_{3, 4, 5}} &= \frac{K_{2345, 4512}}{-K_{2, 4, 5}} = \frac{K_{2345, 5123}}{K_{2, 3, 5}} \\
 &= \frac{K_{2345, 1234}}{-K_{2, 3, 4}} = \frac{K_{3451, 4512}}{K_{1, 4, 5}} = \dots = k_{III}^{(5)}.
 \end{aligned}$$

When f_i ($i=1, 2, 3, 4, 5$) are quartic forms, $k_I^{(5)}$, $k_{II}^{(5)}$ and $k_{III}^{(5)}$ become constant. If 5 binary quartic forms be referred to the *normal curve in space of 4 dimensions*, (8) is equivalent to the following theorem, which is *self-evident*: The polar pentahedron of the polar pentahedron of a given pentahedron, with respect to a fixed space quartic (the normal curve), coincides with the given pentahedron.

These results can be easily extended to the systems of 6, 7,, or $n+1$ forms respectively.

Takedao, January 8, 1918.

On the Algebraic Correspondence,

by

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In this note we shall prove the theorem: *all point-transformations between two planes by which all the straight lines on one plane will be transformed into algebraic curves of order n on the other plane are necessarily algebraic.*

Let

$$x=f(\xi, \eta),$$

$$y=g(\xi, \eta)$$

be the equations defining the correspondence, where f and g are supposed to be analytic but not necessarily uniform. Take any two points (a, b) , (c, d) on the (ξ, η) plane and consider two systems of straight lines

$$\eta-b=\kappa(\xi-a), \quad (1)$$

$$\eta-d=\lambda(\xi-c), \quad (2)$$

where κ and λ are parameters.

Let x_1, y_1 be one of the points corresponding to (a, b) and x_2, y_2 be one of the points corresponding to (c, d) . Then, for the algebraic curve C_κ corresponding to (1), we have at the point (x_1, y_1)

$$\left(\frac{dy}{dx}\right)_1 = \frac{g_\xi(a, b) + g_\eta(a, b)\kappa}{f_\xi(a, b) + f_\eta(a, b)\kappa},$$

$$\begin{aligned} \left(\frac{d^2y}{dx^2}\right)_1 = & \left[\{g_{\xi\xi}(a, b) + 2g_{\xi\eta}(a, b)\kappa + g_{\eta\eta}(a, b)\kappa^2\} \{f_\xi(a, b) + f_\eta(a, b)\kappa\} \right. \\ & \left. - \{f_{\xi\xi}(a, b) + 2\kappa f_{\xi\eta}(a, b) + f_{\eta\eta}(a, b)\kappa^2\} \{g_\xi(a, b) + g_\eta(a, b)\kappa\} \right] \\ & \div \{f_\xi(a, b) + \kappa f_\eta(a, b)\}^3 \end{aligned}$$

.....

Similarly for the algebraic curve C_λ corresponding to (2) we have

$$\left(\frac{dy}{dx}\right)_2 = \frac{g_{\epsilon}(c, d) + g_{\eta}(c, d)\lambda}{f_{\epsilon}(c, d) + f_{\eta}(c, d)\lambda},$$

$$\left(\frac{d^2y}{dx^2}\right)_2 = [\{g_{\epsilon\epsilon}(a, b) + 2g_{\epsilon\eta}(a, b)\lambda + g_{\eta\eta}(a, b)\lambda^2\} \{f_{\epsilon}(a, b) + f_{\eta}(a, b)\lambda\} \\ - \{f_{\epsilon\epsilon}(c, d) + 2f_{\epsilon\eta}(c, d)\lambda + f_{\eta\eta}(c, d)\lambda^2\} \{g_{\epsilon}(a, b) + g_{\eta}(a, b)\lambda\}] \\ \div \{f_{\epsilon}(a, b) + \lambda f_{\eta}(a, b)\}^3,$$

.....

Now the equation of an algebraic curve of order n will algebraically be found in terms of a certain number of differential coefficients of the curve at a point (in general $\varphi(n) - 1 = \frac{n(n+3)}{2} - 1$ differential coefficients of the curve).

Two cases may arise: namely the case where after a certain differential coefficient $\frac{d^k y}{dx^k}$ the above expressions on the right hand sides become indeterminate for every value of (a, b) and (c, d) , in which case f and g are readily shown to be rational integral functions of x and y . Otherwise we can find (a, b) , (c, d) so that the above expression will not be indeterminate. Then the rational integral equations to C_{κ} and C_{λ} are found to be

$$F(x, y, \kappa) = 0,$$

$$G(x, y, \lambda) = 0,$$

where F and G are rational and integral in x and y , and algebraic in κ and λ . These equations may be looked upon as the equations defining the correspondence.

Now κ and λ can be replaced respectively by

$$\frac{\eta - b}{\xi - a} \quad \text{and} \quad \frac{\eta - d}{\xi - c}$$

and consequently x and y are algebraic functions of ξ and η . Thus the theorem in consideration is established.

Let us next propose the following problem: to determine all point-transformations between two planes by which all the straight lines on one plane will be transformed into circles (or straight lines) on the other plane. This problem has been already solved by G. Scheffers in the *Leipziger Berichte*, 1898.

We shall solve this problem by using the above mentioned theorem. The transformations in consideration must be necessarily algebraic by the above theorem and must be necessarily $(1, 1)$, $(1, 2)$ or $(2, 2)$ correspondences.

They cannot evidently be $(2, 2)$ correspondences⁽¹⁾ and consequently they must be contained in $(1, 2)$ quadratic correspondences of de Paoli and H. Liebmann⁽²⁾, or in circular transformations.

All $(1, 2)$ quadratic transformations can, as is known, be transformed into the following form by proper composition of certain collineations: Take two points I, J and a conic passing through I and J ; draw two tangents at the points I and J to the conic and let O be the point of intersection. Take a point P on the plane and find its polar p with respect to the conic and let the points of intersection of p with the conic be X_1, X_2 and join $X_1 I, X_2 J$ and $X_1 J, X_2 I$, so that $X_1 I, X_2 J$ intersect in $Q_1, X_1 J, X_2 I$ in Q_2 . If we let correlate the point P to the points Q_1, Q_2 , which are collinear with O , we get a $(1, 2)$ quadratic correspondence.

In our case I and J must be necessarily the two imaginary circular points at infinity, and consequently the conic must be a circle. Two cases may be distinguished, according as the radius of the circle is real or purely imaginary. All circular transformations can, as is known, be transformed, by proper composition of certain collineations, into inversions. Thus we have the following three possible reduced forms:

1. Liebmann's transformation with a real fundamental circle.
2. Liebmann's transformation with an imaginary fundamental circle.
3. Inversion.

If we wish possibly to avoid imaginary operations, we proceed as follows: passing through the point P draw any chord AB of the fundamental circle; also draw the circle \mathfrak{N} passing through A, B cutting the fundamental circle orthogonally. The line joining the point P to the centre of the fundamental circle cuts the circle \mathfrak{N} in points Q_1, Q_2 , which are corresponding points of P . In the case where the fundamental circle is imaginary, the above method of construction requires slight modifications.

⁽¹⁾ T. Kubota, Über $(2, 2)$ deutige quadratische Verwandtschaften, Science Reports of the Tôhoku Imperial University, 1918.

⁽²⁾ H. Liebmann, Dissertation in Jena, 1895.

It would be especially important to notice here that the above method of solution can immediately be extended to the space of three dimensions while it seems to be impossible to extend Scheffer's method to three dimensions.

January 1918.

Un théorème sur les continus,

par

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Le but de cette note est la démonstration du théorème suivant :

Théorème. *Un ensemble de points dans l'espace à m dimensions, borné et fermé, qui ne peut être décomposé en deux ensembles fermés sans points communs, ne peut être aussi décomposé en une infinité dénombrable d'ensembles fermés sans points communs deux à deux.*

Démonstration. Soit P un ensemble de points donné dans l'espace euclidien à m dimensions, borné et fermé, ne pouvant être décomposé en deux ensembles fermés sans points communs, et supposons que P est une somme d'une infinité dénombrable d'ensembles fermés sans points communs deux à deux :

$$P = P_1 + P_2 + P_3 + \dots$$

Il s'ensuit immédiatement de nos hypothèses que l'ensemble P est un continu (cantorien). Pour tous deux points p_1 et p_2 de P et pour tout nombre positif ε existe donc une suite finie de points $p', p'', \dots, p^{(k)}$, appartenant à P et tels que chacune des distances $p_1 p', p' p'', p'' p''', \dots, p^{(k)} p_2$ est $< \varepsilon$. Nous appellerons une telle suite *chaîne* entre p_1 et p_2 par rapport à ε .

Les ensembles P_1 et P_2 étant bornés (comme parties de P) et fermés, la distance de P_1 à P_2 est positive : désignons-la par 2δ et considérons l'ensemble Q de tous les points de l'espace m -dimensionnel dont la distance à P_2 est $\leq \delta$. L'ensemble Q sera évidemment fermé, et à l'intérieur de Q existera au moins un point p_2 de l'ensemble P_2 , et à l'extérieur de Q — au moins un point p_1 de P_1 (puisque l'ensemble P_2 est évidemment tout intérieur à Q , et l'ensemble P_1 — tout extérieur à Q).

Nous démontrerons qu'il existe un continu contenant le point p_2 et un point de la frontière de Q et contenu dans P et Q (1).

(1) Cf. S. Janiszewski: Thèse, Paris 1911. Théorème IV (p. 22). La démonstration de M. Janiszewski s'appuie d'ailleurs sur un lemme assez compliqué.

Désignons par C l'ensemble de tous ces points p de l'espace à m dimensions, appartenant à Q et pour lesquels il existe par rapport à tout ε positif une chaîne entre p_2 et p , dont tous les points appartiennent à P et à Q . Je dis que la frontière de Q contient au moins un point de l'ensemble C .

En effet, l'ensemble P étant un continu, il existe par rapport à tout ε positif une chaîne entre p_2 et p_1 : soit $p(\varepsilon)$ ce point de cette chaîne, qui précède le premier point extérieur à Q ; la distance de $p(\varepsilon)$ à la frontière de Q sera évidemment $< \varepsilon$, et il existera une chaîne entre p_2 et $p(\varepsilon)$ par rapport à ε , contenue dans P et dans Q . Considérons maintenant l'ensemble des points $p\left(\frac{1}{n}\right)$ ($n=1, 2, 3, \dots$) tous ces points appartenant à Q , et l'ensemble Q étant borné (comme formé des points dont la distance à l'ensemble borné P_2 est $\leq \delta$), l'ensemble des points $p\left(\frac{1}{n}\right)$ sera aussi borné, et par suite il existe une suite croissante d'indices n_1, n_2, n_3, \dots , telle que la suite des points $p\left(\frac{1}{n_k}\right)$ ($k=1, 2, 3, \dots$) est convergente. Désignons par p_0 sa limite: le point $p\left(\frac{1}{n}\right)$ ayant une distance $< \frac{1}{n}$ de la frontière de Q , le point p_0 appartiendra évidemment à la frontière de Q . Or, le point p_0 étant limite des points $p\left(\frac{1}{n_k}\right)$, et entre p_2 et $p\left(\frac{1}{n_k}\right)$ existant une chaîne par rapport à $\frac{1}{n_k}$ contenue dans P et dans Q , nous concluons (d'après $\lim_{k \rightarrow \infty} n_k = \infty$) que le point p_0 appartient à C .

On voit sans peine que l'ensemble C est fermé et qu'il contient le point p_2 .

Or, je dis que l'ensemble C ne peut être décomposé en une somme de deux ensembles fermés sans points communs. En effet, admettons qu'il existe une décomposition $C=A+B$, où A et B sont des ensembles fermés sans points communs, et supposons, par exemple, que A contient le point p_2 . Désignons par $2d$ la distance entre les ensembles (évidemment bornés) A et B , et par R —l'ensemble de tous les points de l'espace à m dimensions qui ont une distance $\leq d$ de A . Soit b un point donné quelconque de B . Le point b appartenant à C , il existe, par rapport à tout ε positif une chaîne entre p_2 et b , contenue dans P et Q : comme plus haut, nous en concluons sans peine qu'il existe pour tout ε positif

un point de Q ayant une distance $< \varepsilon$ de la frontière de R , d'où résulte ensuite que la frontière de R contient au moins un point de l'ensemble C . Ce point ne pourrait évidemment appartenir ni à A ni à B (ayant une distance $\geq d$ de chacun de ces ensembles), c'est ce qui est impossible, d'après $C=A+B$. Nous avons donc démontré que l'ensemble C ne peut être décomposé en une somme de deux ensembles fermés sans points communs. Or, l'ensemble C étant fermé et contenant plus qu'un point (puisqu'il contient p_0 et p_2), nous en concluons que C est un continu.

L'ensemble C est évidemment un sous-ensemble de $P=P_1+P_2+P_3+\dots$ (puisque tout point de C est, comme on voit sans peine, point d'accumulation de l'ensemble fermé P): nous pouvons donc poser :

$$C=CP_1+CP_2+CP_3+\dots, \quad (1)$$

où les termes du côté droit sont des ensembles fermés ou vides, sans points communs deux à deux. Or, l'ensemble CP_2 est non vide, puisqu'il contient toutefois le point p_2 . Si tous les autres termes du côté droit de (1) seraient vides, nous aurions $C=CP_2$, ce qui est impossible, puisque, comme nous savons, l'ensemble C contient le point p_0 qui n'appartient pas à P_2 . Or, l'ensemble C étant un continu, il est impossible que C soit une somme d'un nombre fini ≥ 2 d'ensembles fermés sans points communs deux à deux : il s'en suit que le côté droit de (1) contient nécessairement une infinité de termes qui sont des ensembles non vides.

La formule (1) fournit donc une décomposition :

$$C=F_1+F_2+F_3+\dots,$$

où F_1, F_2, F_3, \dots sont des ensembles fermés sans points communs deux à deux (F_1, F_2, F_3, \dots sont ici des termes non vides consécutifs du développement (1)). De plus, l'ensemble C , qui ne contient aucun point extérieur de l'ensemble Q , ne contiendra aucun point de P_1 .

Pour les ensembles C, F_1 et F_2 nous pouvons répéter le même raisonnement que nous avons fait pour les ensembles P, P_1 et P_2 , ce qui donnera un nouveau développement

$$C_1=G_1+G_2+G_3+\dots,$$

où C_1 est un continu contenu dans C , sans points communs avec F_1 et G_1, G_2, G_3, \dots sont des ensembles fermés sans points communs deux à deux.

En répétant notre raisonnement pour les ensembles C_1, G_1 et G_2 , nous trouvons un nouveau développement

$$C_2 = H_1 + H_2 + H_3 + \dots,$$

où C_2 est un continu sans points communs avec G_1 , contenu dans C_1 , et ainsi de suite.

On voit sans peine que le continu C_n sera sans points communs avec $P_1 + P_2 + \dots + P_n + P_{n+1}$.

C, C_1, C_2, C_3, \dots étant une suite infinie d'ensembles fermés dont chacun est contenu dans le précédent, il existe au moins un point p communs à tous les ensembles C_n ($n=1, 2, \dots$). Le point p serait donc un point de l'ensemble P (dont les C_n sont sous-ensembles); or, C_n ne contenant aucun point de $P_1 + P_2 + \dots + P_{n+1}$, le point p ne peut pas appartenir à $P = P_1 + P_2 + \dots$, c'est ce qui implique une contradiction.

Notre théorème est donc démontré.

Remarquons que pour les ensembles fermés, mais non bornés, notre théorème ne subsiste pas en général. On peut p. e. construire sans peine un ensemble fermé de points dans l'espace à trois dimensions qui ne peut être décomposé en deux ensembles fermés sans points communs, mais qui peut être décomposé en une infinité dénombrable d'ensembles fermés sans points communs deux à deux. Or, on pourrait démontrer sans peine que dans l'espace à *une* dimension notre théorème reste vrai même pour les ensembles fermés non bornés. Il serait intéressant d'examiner si notre théorème est vrai pour tous les ensembles fermés dans le plan.

Repeated Solutions of a Certain Class of Linear Functional Equations,

by

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1. In a fundamental system of solutions of a linear differential equation it is clear that no given solution is repeated; for, if so, the set would not be linearly independent. Hence, if we are to have in linear differential equations the analogue of multiple solutions of algebraic equations in one variable it must be through some modification of the definition of multiple solution so that a solution need not be repeated unchanged in that fundamental system of solutions so as to meet the requirements of that definition.

For the linear differential equation

$$a_0 D^n y + a_1 D^{n-1} y + \dots + a_{n-1} Dy + a_n y = 0, \quad (1)$$

in which D denotes d/dx and the a 's are given functions of x , we shall say that y_1 is a *solution of order r* , or an *r -fold solution*, if not only y_1 but also $x y_1, x^2 y_1, \dots, x^{r-1} y_1$ are solutions of (1) while $x^r y_1$ is not a solution of (1). We shall say that a solution y_1 of (1) is *simple* when $x y_1$ is not a solution of (1). A solution which is not simple will be called a *repeated solution*.

Let y_1 be a repeated solution of (1). Then $y = x y_1$ also satisfies this equation. Substituting $x y_1$ for y in (1) and reducing by aid of the fact that y_1 itself is a solution of (1) we find that y_1 must also satisfy the equation

$$n a_0 D^{n-1} y + (n-1) a_1 D^{n-2} y + \dots + a_{n-1} y = 0.$$

Therefore, if y_1 is a repeated solution of (1) it also satisfies the equation obtained from (1) by formal differentiation with respect to D . This will be recognized as analogous to a corresponding theorem in the theory of algebraic equations.

If y_1 is an r -fold solution of (1) it is now clear that it must satisfy each of the equations obtained by successive formal differentiation of (1) $r-1$ times with respect to D .

With reference to the converse of this result, let us suppose that y_1 satisfies (1) and the $r-1$ equations obtained from (1) by $r-1$ successive formal differentiations with respect to D . Then it is easy to show that $xy_1, \dots, x^{r-1}y_1$ are also solutions of (1), so that y_1 itself is an r -fold solution of (1).

From these results it is apparent that the theory of multiple solution of homogeneous linear differential equations must have many analogies with that of multiple solutions of algebraic equations. It is also clear that relatively few of the properties of the symbol D will be needed in developing the elements of this theory. This suggests a postulational treatment based on certain properties of the symbol D which it has in common with other functional operators. In this note we develop the first elements of such a theory for a functional operator D including as special cases the operator d/dx of the infinitesimal calculus and the operators E and E_1 of the difference calculus, namely,

$$Ef(x) = f(x+1), \quad E_1f(x) = f(qx),$$

where in the last q is a constant different from unity in absolute value.

On first reading one who prefers to do so may treat the postulates of the next section as theorems about the operator D , $\equiv d/dx$, and think of the discussion as applying differential equations alone.

2. We are to consider multiple solutions of homogeneous linear functional equations of the form

$$a_0(x)D^n y + a_1(x)D^{n-1}y + \dots + a_{n-1}(x)Dy + a_n(x)y = 0 \quad (2)$$

in which the a 's are given functions of x and the operator D possesses properties which may be described as follows:

A. If u and v are any two functions of x then $D(u+v) = Du + Dv$.

B. There exists a class $\{C(x)\}$ of functions $C(x)$ of x such that for any $C(x)$ of this class and every function $u(x)$ we have

$$D\{C(x)u(x)\} = C(x)D\{u(x)\}.$$

We suppose moreover that a function $\bar{C}(x)$ satisfying the relation $D\{\bar{C}(x)u(x)\} = \bar{C}(x)D\{u(x)\}$ for a given non-vanishing function $u(x)$ belongs to the class $\{C(x)\}$.

For $D \equiv d/dx$ these functions $C(x)$ are arbitrary constants; for $D \equiv E$ they are arbitrary periodic functions of period unity; for $D \equiv E_1$ they

are arbitrary q -periodic functions, that is, functions such that $C(qx) = C(x)$.

C. The class of functions $\{C(x)\}$ contains the particular functions 0 and 1.

D. The general solution of (2) is of the form

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n,$$

where C_1, C_2, \dots, C_n are arbitrary functions of class $C(x)$ of postulate B and the y 's are such that the determinant

$$G(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ Dy_1 & Dy_2 & \cdots & Dy_n \\ D^2y_1 & D^2y_2 & \cdots & D^2y_n \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-1}y_1 & D^{n-1}y_2 & \cdots & D^{n-1}y_n \end{vmatrix}$$

is not identically zero.

E. There exists a function $t(x)$ of x such that for every function $u(x)$ of x the value of $Du(x)$ at a point $x=a$ depends solely on the values of $u(x)$ in the neighborhood of the point $x=t(a)$.

We denote by $s(x)$ the inverse of this function. For $D \equiv d/dx$, E , E_1 , we have $t(x) = x$, $x+1$, qx respectively. The corresponding values of $s(x)$ are x , $x-1$, x/q .

F. There exists a function $p(x)$ of x such that for every function $u(x)$ of x and every non-negative integer k we have

$$D^k \{ p(x) u(x) \} = p(x) D^k u(x) + k D^{k-1} u \{ t(x) \}.$$

For the values d/dx , E , E_1 of D we have for p the functions x , x , $\log x / \log q$ respectively.

G. Operators H , F , F_1 exist such that for every pair of functions u and v of x and every non-negative integer k we have

$$\begin{aligned} D \{ u(x) D^k v(x) \} &= u(x) D^{k+1} \{ v(x) \} + H \{ u(x) \} \cdot D^k [v \{ t(x) \}] \\ &= F u \cdot D^{k+1} v + F_1 u \cdot D^k v. \end{aligned}$$

Moreover, if $HC=0$, then it is necessary and sufficient that C shall be of the class $\{C(x)\}$.

For $D \equiv d/dx$, E , E_1 , we have $H = d/dx$, Δ , δ , where

$$\Delta f(x) = f(x+1) - f(x), \quad \partial f(x) = f(qx) - f(x).$$

Also, for $D \equiv d/dx$ we have $F=1$, $F_1=D$; for $D \equiv E$ we have $F=E$, $F_1=0$; for $D=E_1$ we have $F=E_1$, $F_1=0$. [The operator 1 turns every function into itself; the operator 0 turns every function into 0.]

3. We shall say that a solution y_1 of (2) is *simple* if py_1 is not a solution of (2). If both y_1 and py_1 are solutions of (2) we shall say that y_1 is a *repeated solution* of (2). If $y_1, py_1, p^2 y_1, \dots, p^{r-1} y_1$ are all solutions of (2), but $p^r y_1$ is not a solution of (2), we shall say that y_1 is an *r-fold solution* of (2), or a *solution of (2) of order r*.

Let y_1 be a repeated solution of (2). For y in (2) substitute the solution py_1 of (2) and reduce by means of (2) itself and postulate F . Thus we have

$$na_0(x) D^{n-1} y_1 \{t(x)\} + (n-1) a_1(x) D^{n-1} y_1 \{t(x)\} \\ + \dots + a_{n-1}(x) y_1 \{t(x)\} = 0. \quad (3)$$

In this equation replace x by $s(x)$ and drop the subscript from y . Then we have

$$na_0\{s(x)\} D^{n-1} y(x) + (n-1) a_1\{s(x)\} D^{n-2} y(x) \\ + \dots + a_{n-1}\{s(x)\} y(x) = 0. \quad (4)$$

We observe that this equation may be obtained from (2) by formal differentiation with respect to D and the replacing of x by $s(x)$ in the *coefficients* of the resulting equation. A repeated solution y_1 of (2) is clearly also a solution of (4).

Again, let y_1 be a common solution of (2) and (4). Then y_1 satisfies (3). Add (3) to the equation obtained from (2) on multiplying by p and replacing y by y_1 . Making use now of postulate F we see that py_1 also satisfies (2).

Thus we have the following theorem:

Theorem. *A necessary and sufficient condition that a function y_1 shall be a repeated solution of (2) is that it shall satisfy both equation (2) and the equation obtained from (2) by formal differentiation with respect to D and the replacing of x by $s(x)$ in the coefficients of the resulting equation.*

The equation thus gotten from (2) will be called the (*first*) *derived equation* of (2). The (first) derived equation of the latter will be called the second derived equation of (2); and so on.

An application of the preceding theorem leads immediately to the following general result:

Theorem. *A necessary and sufficient condition that a function y_1 shall be an r -fold solution of (2) is that it shall satisfy (2) and the first $r-1$ derived equations and shall not satisfy the r^{th} derived equation.*

4. One naturally raises the question as to whether there is the same intimate connection between repeated solutions of (2) and repeated factors of its first member as holds in the case of algebraic equations. That the question is to be answered in the negative is readily shown by means of an example. Let us consider the differential equation

$$\left(\frac{d}{dx} + x\right) \cdot \left(\frac{d}{dx} + x\right) \cdot \frac{d}{dx} \cdot y = 0.$$

This reduces to

$$\frac{d^3 y}{dx^3} + 2x \frac{d^2 y}{dx^2} + (x^2 + 1) \frac{dy}{dx} = 0.$$

If y is a repeated solution of this equation it must also satisfy the equation

$$3 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + (x^2 + 1)y = 0;$$

and hence also the equation obtained from this one by differentiation with respect to x , namely:

$$3 \frac{d^3 y}{dx^3} + 4x \frac{d^2 y}{dx^2} + (x^2 + 5) \frac{dy}{dx} + 2xy = 0.$$

Multiplying the last three equations in order by -9 , $2x$, 3 and adding we have a first order equation which y must satisfy:

$$(2x^2 + 6) \frac{dy}{dx} + (2x^3 + 8x)y = 0.$$

Hence if our differential equation has a repeated solution it must be

$$y = C e^{-x^2/2} (x^2 + 3)^{-\frac{1}{2}}.$$

But this does not satisfy the second order equation above. Hence the differential equation in consideration has no repeated solution even though its first member may be factored so as to exhibit a repeated factor.

Again, the equation

$$x^2 \frac{d^4 y}{dx^4} - 2x \frac{d^3 y}{dx^3} + 2 \frac{d^2 y}{dx^2} = 0$$

has the fundamental system of solutions $1, x, x^3, x^4$, containing the two repeated solutions $1, x$. The equation may be written in factored form as

$$\left(x \frac{d}{dx} - 2\right) \cdot \left(x \frac{d}{dx} - 1\right) \cdot \frac{d}{dx} \cdot \frac{d}{dx} \cdot y = 0.$$

In this form one factor is repeated corresponding to the repeated solution 1 . But the other factor is not repeated. One may also put the equation in the form

$$\left(\frac{d}{dx} - \frac{1}{x}\right) \cdot \left(\frac{d}{dx} - \frac{1}{x}\right) \cdot \frac{d}{dx} \cdot \frac{d}{dx} \cdot y = 0$$

so that there is a repeated factor corresponding to each repeated solution. Or, again, one may break the equation into factors in the form

$$\left(\frac{d}{dx} - \frac{1}{x}\right) \cdot \left(\frac{d}{dx} + \frac{1}{x}\right) \cdot \left(\frac{d}{dx} - \frac{2}{x}\right) \cdot \frac{d}{dx} \cdot y = 0,$$

in which no factor is repeated.

From considerations of this sort it is clear that there is no such suitable connection between factorization and repeated solutions as would make it convenient to investigate repeated solutions by aid of repeated factors in some one factored form of the equation.

Similar remarks may obviously be made about difference and q -difference equations.

5. It is obvious that a differential equation of the second order with a repeated solution may be solved by quadratures; for by the first theorem of section 3 the repeated solution y_1 satisfies an equation of the first order. Then y_1 and xy_1 form a fundamental system of solutions of the given equation.

In general, let us suppose that equation (2) of order n ($n > 2$) has a repeated solution y_1 . Then y_1 satisfies (2) and (4). On account of postulate G one may operate one equation (4) with D and obtain a resulting equation of the form (2). Between this equation and (2) and (4) one may in general eliminate D^n and D^{n-1} and thus obtain an equation of order $n-2$ which is satisfied by y_1 .

Let us apply these considerations to a differential equation of the third order with a repeated solution:

$$a_0 \frac{d^3 y}{dx^3} + a_1 \frac{d^2 y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = 0.$$

The repeated solution also satisfies the equations

$$3a_0 \frac{d^2 y}{dx^2} + 2a_1 \frac{dy}{dx} + a_2 y = 0,$$

$$3a_0 \frac{d^3 y}{dx^3} + \left(3 \frac{da_0}{dx} + 2a_1\right) \frac{d^2 y}{dx^2} + \left(2 \frac{da_1}{dx} + a_2\right) \frac{dy}{dx} + \left(\frac{da_2}{dx}\right) y = 0.$$

We may eliminate from the last three equations the second and third derivatives of y with respect to x and thus obtain the equation

$$\begin{aligned} &\left(2a_1^2 - 6a_0 a_2 + 6a_0 \frac{da_1}{dx} - 6a_1 \frac{da_0}{dx}\right) \frac{dy}{dx} \\ &+ \left(a_1 a_2 - 9a_0 a_3 + 3a_0 \frac{da_2}{dx} - 3a_2 \frac{da_0}{dx}\right) y = 0. \end{aligned}$$

Two cases arise which we treat separately.

If the coefficient of dy/dx in the last equation is not identically zero we have a first order equation satisfied by y_1 . It can be solved by quadratures. Then y_1 and xy_1 are two linearly independent solutions of the given equation of the third order. An additional solution in a fundamental system of solutions may then be found by quadratures. Hence in this case the general solution of the given equation may be obtained by quadratures.

Suppose next that the coefficient of dy/dx in the last equation is identically zero. This is just the condition that the second order equation above has a repeated solution (see section 6 below). Such solution y_1 may be found by quadratures. Then y_1 , xy_1 , $x^2 y_1$ form a fundamental system of solutions of the given equation of the third order.

Hence we have the following theorem :

Theorem. *If a homogeneous linear differential equation of the third order has a repeated solution the general solution of the equation may be obtained by quadratures.*

In a similar way one may prove the like theorem for difference and q -difference equations.

It is easy to see furthermore that such an equation of order n may be solved by quadratures in case it possesses an n -fold or an $(n-1)$ -fold solution.

6. We shall now obtain, in terms of the coefficients alone, a necessary and sufficient condition that equation (2) shall possess a repeated solution. For this purpose let us consider the equations obtained from (2) and (4) through operating upon them repeatedly with D . The repeated solution of (2) must satisfy every equation so obtained.

In view of postulate G it is clear that the equations resulting from (2) by operating with D^k , $k=0, 1, \dots, n-2$, may be written in the form

$$\alpha_{0,k} D^{n+k} y + \alpha_{1,k} D^{n+k-1} y + \dots + \alpha_{n+k,k} y = 0, \quad k=0, 1, \dots, n-2, \quad (5)$$

where the coefficients $\alpha_{i,j}$ are functions of x which are expressible in terms of the coefficients a_0, a_1, \dots, a_n of equation (2). Similarly those equations resulting from (4) by operating with D^k , $k=0, 1, \dots, n-1$, may be written in the form

$$b_{0,k} D^{n+k-1} y + b_{1,k} D^{n+k-2} y + \dots + b_{n+k-1,k} y = 0, \quad k=0, 1, \dots, n-1, \quad (6)$$

where the $b_{i,j}$ are functions of x which are expressible in terms of a_0, a_1, \dots, a_n . We may look upon the $2n-1$ equations (5) and (6) as consistent homogeneous algebraic equations involving the $2n-1$ unknown quantities $y, Dy, D^2y, \dots, D^{2n-2}y$. The determinant of these equations must vanish identically. Hence a necessary condition that (2) shall have a repeated solution is that the determinant $\Delta(x)$ shall be identically zero, where

$$\Delta(x) = \begin{vmatrix} \alpha_{0,n-2} & \alpha_{1,n-2} & \dots & \alpha_{k,n-2} & \alpha_{k+1,n-2} & \dots & \alpha_{2n-2,n-2} \\ 0 & \alpha_{0,n-3} & \dots & \alpha_{k-1,n-3} & \alpha_{k,n-3} & \dots & \alpha_{2n-3,n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_{0,0} & \alpha_{1,0} & \dots & \alpha_{n,0} \\ 0 & 0 & \dots & 0 & b_{0,0} & \dots & b_{n-1,0} \\ 0 & 0 & \dots & b_{0,1} & b_{1,1} & \dots & b_{n,1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & b_{0,n-2} & \dots & b_{k-1,n-2} & b_{k,n-2} & \dots & b_{2n-3,n-2} \\ b_{0,n-1} & b_{1,n-1} & \dots & b_{k,n-1} & b_{k+1,n-1} & \dots & b_{2n-2,n-2} \end{vmatrix}. \quad (7)$$

For the differential equation $a_0 y'' + a_1 y' + a_2 y = 0$ this condition reduces to

$$a_1^2 - 4 a_0 a_2 + 2 a_0 a_1' - 2 a_0' a_1 \equiv 0,$$

since $a_0 \neq 0$. For the differential equation $y''' + a_2 y' + a_3 y = 0$ this condition is

$$4 a_2^3 + 6 a_2 a_2'' - 3 a_2'^2 - 18 a_2 a_3' + 27 a_3^2 \equiv 0.$$

We may replace the determinant $\Delta(x)$ of order $2n-1$ by a deter-

minant of order n which defines the same function of x as $\Delta(x)$ itself. For this purpose write equation (4) in the form

$$P_0(y) \equiv C_{00} D^{n-1} y + C_{10} D^{n-2} y + \dots + C_{n-1,0} y = 0. \quad (8)$$

Operate upon this with D and reduce the order of the resulting equation by adding to it some multiple of equation (2); thus we have

$$P_1(y) \equiv C_{01} D^{n-1} y + C_{11} D^{n-2} y + \dots + C_{n-1,1} y = 0. \quad (9)$$

Operate again in a similar manner and reduce the equation similarly to

$$P_2(y) \equiv C_{02} D^{n-1} y + C_{12} D^{n-2} y + \dots + C_{n-1,2} y = 0. \quad (10)$$

We may continue this process until we have a system of n equations. Then it is clear that the functions $\Delta(x)$ and $M(x)$ are identical, where $M(x)$ is the determinant of this system, namely:

$$M(x) = \begin{vmatrix} C_{00} & C_{10} & C_{20} & \dots & C_{n-1,0} \\ C_{01} & C_{11} & C_{21} & \dots & C_{n-1,1} \\ \dots & \dots & \dots & \dots & \dots \\ C_{0,n-1} & C_{1,n-1} & C_{2,n-1} & \dots & C_{n-1,n-1} \end{vmatrix}. \quad (11)$$

A necessary condition that (2) shall have a repeated solution is that $\Delta(x)$, or $M(x)$ shall vanish identically. We shall now show that this is also a sufficient condition.

Accordingly, we suppose that $M(x)$ vanishes identically. Let y_1, y_2, \dots, y_n be a fundamental system of solutions of (2). Then not all the quantities

$$P_0(y_1), P_0(y_2), \dots, P_0(y_n)$$

vanish identically, where $P_0(y)$ is defined as in equation (8); for, otherwise, the determinant $G(x)$ in postulate D would vanish identically contrary to the hypothesis that y_1, y_2, \dots, y_n form a fundamental system of solutions of (2).

Consider the system of equations

$$\sum_{i=1}^n C_i P_j(y_i) = 0, \quad j=0, 1, \dots, n-1, \quad (12)$$

involving the unknown quantities C_i , the P 's being defined as in equation (8) and following. The determinant of this system obviously factors into $G(x)M(x)$; it is therefore identically zero, while its rank is not zero. Hence it suffices for determining functions C_1, \dots, C_n of x . For the arbitrary

element or elements in this solution of (12) we take functions belonging to the class $\{C(x)\}$ of postulate B . By the use of the latter part of postulate G it may readily be shown then that each C_i is of class $\{C(x)\}$.

With these values of C_i form the solution

$$\bar{y} = \sum_{i=1}^n C_i y_i$$

of equation (2). In order that it shall be a repeated solution it is necessary and sufficient that $p\bar{y}$ shall be a solution of (2). If we substitute $p\bar{y}$ for y in the first member of (2) and reduce by means of the fact that y_1, y_2, \dots, y_n are solutions of (2) we obtain as a result the first member of the first equation (12). This has the value zero. Hence $p\bar{y}$ is a solution of (2); and therefore \bar{y} is a repeated solution of (2).

Thus we have the following theorem:

Fundamental Theorem. *A necessary and sufficient condition that equation (2) shall have a repeated solution is that the determinant $\Delta(x)$ in (7) [or the determinant $M(x)$ in (11)] shall vanish identically.*

If equation (2) is written so that a_0 is unity we may call $\Delta(x)$ the *discriminant* of (2). It is clear that the relation between $\Delta(x)$ and (2) is closely analogous to that between an algebraic equation and its discriminant.

December 1917.

Group-Theory Proof of Two Elementary Theorems in Number Theory,

by

G. A. MILLER, Urbana, Ill., U.S.A.

Let s and t represent two group operators such that all the conjugates of s under the different powers of t are commutative with each other. If we employ the notation

$$t s t^{-1} = s_1 s, \quad t s_1 t^{-1} = s_2 s_1, \quad \dots, \quad t s_a t^{-1} = s_{a+1} s_a,$$

a being any positive integer, it results that

$$t^n s t^{-n} = s_n s_{n-1}^{(n)} \dots s_{n-r}^{(n)} \dots s_1^n s,$$

where $r=1, 2, \dots, n-1$. Hence it follows that

$$\begin{aligned} (t s)^n &= t s t^{-1} \cdot t^2 s t^{-2} \cdot \dots \cdot t^{n-1} s t^{1-n} \cdot t^n s \\ &= s_1^{1+2+\dots+(n-1)} \dots s_{n-1}^{n-1} t^n s \end{aligned}$$

The exponents of s_1, s_2, \dots, s_{n-1} in the last expression are the sums of $n-1, n-2, \dots, 1$ figurate numbers of orders $2, 3, \dots, n$ respectively⁽¹⁾. When n is a prime number p each of these exponents except the last is known to be divisible by p . We proceed to prove this elementary theorem relating to figurate numbers by means of the properties of a very elementary substitution group and thus exhibit additional contact between group theory and number theory.

Let G represent the Sylow subgroup of order p^{v+1} contained in the symmetric group of degree p^2 . If s represents a cyclic substitution of order p contained in G while t is of order p and of degree p^2 and also non-commutative with s it results directly that G is generated by s and t . The subgroup H generated by the conjugates of s under the different powers of t is abelian and of type $(1, 1, 1, \dots)$. Its index under G

(1) Cf. G. S. Carr, *A Synopsis of Elementary Results in Pure Mathematics*, 1886, p. 96.

is p . The central of G is generated by s_{p-1} ⁽¹⁾ and contains the p th power of every substitution found in G .

Since $(ts)^p$ is generated by s_{p-1} it results that the exponent of each of the operators s_1, s_2, \dots, s_{p-2} is $\equiv 0, \text{ mod. } p$. In fact, if one of these exponents were not $\equiv 0, \text{ mod. } p$, we may suppose that $s_\beta, \beta < p-1$, is the first one which satisfies this condition. As all the operators $s_{\beta+1}, \dots, s_{p-1}$ generate a group which does not include s_β it would result from this that t would not be commutative with $(ts)^p$. This is contrary to the known fact that this power is found in the central of G and hence the hypothesis leads to a contradiction.

From the fact that $(ts)^p$ is generated by s_{p-1} it follows therefore that the sums of $p-1, p-2, \dots, 2$ figurate numbers of orders 2, 3, $\dots, p-1$ respectively must be separately $\equiv 0, \text{ mod. } p$, p being any odd prime number. This theorem could clearly be extended by letting n represent powers of prime numbers in the formulas stated above.

A more elementary theorem in number theory which can be directly obtained from the given formulas is as follows: If p represents any prime number then each of the coefficients except the first and last in the expansion of $(a+b)^p$ is divisible by p . In fact, if we let $n=p+1$ in the formula at the end of the first paragraph the exponents of s_1, s_2, \dots, s_{p-1} may be obtained by adding to the preceding exponents of these substitutions the coefficients in order, beginning with the second, of the terms in the expansion of $(a+b)^p$. All of these exponents except that of s_{p-1} must again be $\equiv 0, \text{ mod. } p$, while that of s_{p-1} must be $\equiv 1, \text{ mod. } p$, since $(ts)^{p+1} = t s. s_{p-1}$.

The main element of interest in these developments seems to be due to the fact that they may serve as another illustration of the fundamental concepts involved in group theory. The fact that these very elementary theorems in number theory are by-products of the study of a special substitution group and that they are thus proved anew by considerations which are essentially different from those upon which the well known simple proofs are based seems to merit emphasis.

(1) *American Journal of Mathematics*, vol. 23 (1901), p. 176.

或函數方程式ニ就テ

On a Certain Functional Equation,

林 鶴 一 (仙 臺)

TSURUICHI HAYASHI, Sendai.

窪田教授ガ本誌第十一卷ニ於テ公ニセラレタル論文 Über die periodischen Lösungen in der Variationsrechnung ノ中ニ於テ (同卷第 149 頁ヨリ第 154 頁ニ至ル間參照) 次ノ面白キ定理ノ證明アリ。

代數方程式

$$p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n = 0 \quad (1)$$

ノ根ノ絶對値ガ何レモ 1 ナラザルトキハ⁽¹⁾ 函數方程式

$$p_0 y(x) + p_1 y(x+1) + p_2 y(x+2) + \cdots + p_n y(x+n) = \lambda \quad (2)$$

ハ定數ノ外ニ有理週期 a フ有スル連續的解ヲ有セズ。但シ p_0, p_1, \dots, p_n 及ビ λ ハ所設ノ實數又ハ複素數トス。又 (1) ノ或ル根ノ絶對値ガ 1 ニ等シク其ノ振幅ガ π ノ有理數倍ナルトキニ於テモ a ガ無理數ナラバ函數方程式 (2) ハ定數ノ外ニ a フ週期トスル連續的解ヲ有セズ。

余ガ今此處ニ關與セントスル事項ハ此ノ定理ノ第一部分ニシテ之ニ別證明ヲ與フルト同時ニ、其ノ有效範圍ヲ擴張シ、多少ノ演繹及ビ應用ノ結果ヲ示サントスルニアリ。

1. 函數方程式

$$p_0 y(x) + p_1 y(x+1) + p_2 y(x+2) + \cdots + p_n y(x+n) = \lambda \quad (2)$$

ニ於テ右邊ヲ

$$\begin{aligned} \lambda &= (p_0 + p_1 + p_2 + \cdots + p_n) \frac{\lambda}{p_0 + p_1 + p_2 + \cdots + p_n} \\ &= (p_0 + p_1 + p_2 + \cdots + p_n) \lambda' \end{aligned}$$

(1) 實係數ヲ有スル既約代數方程式ノ總テノ根ノ絶對値ガ 1 ナル爲メノ條件ニ就テハ本誌第 10 卷第 115 頁ニ於ケル Kempner 氏ノ論文ヲ參照スベシ。

ト書キ改メ⁽¹⁾, 之ヲ左邊ニ移セバ

$$p_0\{y(x)-\lambda'\}+p_1\{y(x+1)-\lambda'\}+p_2\{y(x+2)-\lambda'\} \\ +\cdots+p_n\{y(x+n)-\lambda'\}=0$$

ヲ得. 故ニ吾人ハ方程式 (2) ノ代リニ

$$p_0 y(x)+p_1 y(x+1)+p_2 y(x+2)+\cdots+p_n y(x+n)=0 \quad (3)$$

ヲ取り扱ヒテ可ナリ. 此函數方程式ニ有理週期 α ヲ有スル連續的解アリトシ週期 α ヲ $\frac{\alpha}{\beta}$ ニテ表ハシ α ト β トハ正ノ整數ニシテ公約數ナク且ツ α ハ基週期 primitive period ナリトス. 然ラバ α ハ假週期ナリ. α ハ n ヨリ大ナラザルヤモ計リ難シ. 若シ $\alpha \leq n$ ナルトキハ $ka > n$ ナルガ如キ正ノ整數 k ヲ選ベバ ka モ亦假週期ナリ. 今此ノ ka ヲ α ニテ表ハシタリトシ函數方程式 (3) ガ 0 ノ外ニ正ノ整數 a ヲ週期トスル連續的解ヲ有セザル條件ヲ求メントス.

方程式 (3) ヲ

$$p_0 y(x)+p_1 y(x+1)+p_2 y(x+2)+\cdots+p_{a-1} y(x+a-1)=0 \quad (4)$$

トシ添數 v ガ n ヨリ大ナル p_v ハスベテ 0 ナリトス.

方程式 (4) ニ於テ x ノ代リニ $x+1, x+2, \cdots, x+a-1$ ト置ケバ假定ニヨリ α ガ $y(x)$ ノ週期ナルヲ以テ

$$\left. \begin{aligned} p_0 y(x+1)+p_1 y(x+2)+p_2 y(x+3)+\cdots+p_{a-1} y(x) &=0, \\ p_0 y(x+2)+p_1 y(x+3)+p_2 y(x+4)+\cdots+p_{a-1} y(x+1) &=0, \\ \cdots \cdots \cdots & \\ p_0 y(x+a-1)+p_1 y(x)+p_2 y(x+1)+\cdots+p_{a-1} y(x+a-2) &=0 \end{aligned} \right\} \quad (5)$$

ヲ得.

次ギニ二項方程式

$$1-z^a=0$$

ノ根ヲ

$$1, \omega, \omega^2, \omega^3, \cdots, \omega^{a-1} \quad (6)$$

トシ, 其ノ中ノ任意ノ一ツヲ σ ニテ代表セシメ, σ ハ代數方程式

(1) 茲ニ $\lambda \neq 0$ ナルトキ $p_0+p_1+p_2+\cdots+p_n \neq 0$ トノ假定ヲ要ス. 然レドモ $\lambda=0$ トシテ論證スル本節ノ定理ニ $p_0+p_1+p_2+\cdots+p_n \neq 0$ トノ假設ヲナス. 故ニ本節ノ定理ハ $\lambda=0$ ノトキニモ $\lambda \neq 0$ ノトキニモ眞ナリ.

$$p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n = 0 \quad (1)$$

ノ根ナラザリシトス.

方程式 (4) 及 (5) ニ夫々

$$1, \sigma, \sigma^2, \dots, \sigma^{a-1}$$

ヲ乗ジテ相加フレバ

$$\begin{aligned} & p_0 \{ y(x) + \sigma y(x+1) + \sigma^2 y(x+2) + \cdots + \sigma^{a-1} y(x+a-1) \} \\ & + \sigma^{-1} p_1 \{ y(x) + \sigma y(x+1) + \sigma^2 y(x+2) + \cdots + \sigma^{a-1} y(x+a-1) \} \\ & + \sigma^{-2} p_2 \{ y(x) + \sigma y(x+1) + \sigma^2 y(x+2) + \cdots + \sigma^{a-1} y(x+a-1) \} \\ & \dots \dots \dots \\ & + \sigma^{-(a-1)} p_{a-1} \{ y(x) + \sigma y(x+1) + \sigma^2 y(x+2) + \cdots + \sigma^{a-1} y(x+a-1) \} = 0 \end{aligned}$$

即チ

$$\begin{aligned} & \{ p_0 + \sigma^{-1} p_1 + \sigma^{-2} p_2 + \cdots + \sigma^{-(a-1)} p_{a-1} \} \\ & \times \{ y(x) + \sigma y(x+1) + \sigma^2 y(x+2) + \cdots + \sigma^{a-1} y(x+a-1) \} = 0. \end{aligned}$$

然ルニ σ ハ數列 (6) ノ中ニアリテ $z^a - 1 = 0$ ノ根ナリ. 然ルニ σ^{-1} モ亦 $z^a - 1 = 0$ ノ根ナリ. 故ニ數列 (6) ノ中ニアリ. 故ニ σ^{-1} ハ方程式 (1) ノ根ニアラズ. 故ニ

$$p_0 + \sigma^{-1} p_1 + \sigma^{-2} p_2 + \cdots + \sigma^{-(a-1)} p_{a-1} \neq 0.$$

故ニ

$$y(x) + \sigma y(x+1) + \sigma^2 y(x+2) + \cdots + \sigma^{a-1} y(x+a-1) = 0.$$

此ノ中ノ σ ニ數列 (6) ノ一ツヲ順次置換スレバ

$$y(x) + y(x+1) + y(x+2) + \cdots + y(x+a-1) = 0,$$

$$y(x) + \omega y(x+1) + \omega^2 y(x+2) + \cdots + \omega^{a-1} y(x+a-1) = 0,$$

$$y(x) + \omega^2 y(x+1) + \omega^4 y(x+2) + \cdots + \omega^{2(a-1)} y(x+a-1) = 0,$$

$$\dots \dots \dots$$

$$y(x) + \omega^{a-1} y(x+1) + \omega^{2(a-1)} y(x+2) + \cdots + \omega^{(a-1)^2} y(x+a-1) = 0$$

ヲ得. 然ルニ

$$1 + \sigma + \sigma^2 + \cdots + \sigma^{a-1} = 0.$$

故ニ上ノ a 箇ノ方程式ヲ相加フレバ

$$y(x) = 0.$$

故ニ次ノ定理ヲ得.

二ツノ代數方程式

$$1 - x^a = 0,$$

$$p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n = 0, \quad n < a,$$

ガ共通根ヲ有セザルトキハ函數方程式

$$p_0 y(x) + p_1 y(x+1) + p_2 y(x+2) + \cdots + p_n y(x+n) = 0$$

ハ 0 ノ外ニ a ヲ週期トスル連續的解ナシ.

從ツテ又次ノ定理ヲ得.

代數方程式

$$p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n = 0$$

ガ二項方程式

$$1 - x^a = 0, \quad a > n,$$

ノ根ヲ根トセザルトキ函數方程式

$$p_0 y(x) + p_1 y(x+1) + p_2 y(x+2) + \cdots + p_n y(x+n) = \lambda^{(1)}$$

ハ定數ノ外ニ有理數 α/β ヲ週期トスル連續的解ヲ有セス. 但シ β ハ任意ノ正整數ニシテ α ト公約數ヲ有スルコトヲ妨ゲズ.

是レ本論文ノ冒頭ニ掲ゲタル窪田君ノ定理ニシテ其ノ適用ノ範圍ハ遙カニ緩カナリ.

上記ノ證明ニ於テ (4) ガ連續的解ヲ有スレバソハ又 ($\sigma=1$ トスレバ)

$$y(x) + y(x+1) + y(x+2) + \cdots + y(x+a-1) = 0$$

ノ解ナルコトヲ知ル. a ガ偶數ナルトキハ數列 (6) ノ中ニ -1 アルヲ以テ ($\sigma=-1$ トスレバ) 其ノ解ハ又

$$y(x) - y(x+1) + y(x+2) - \cdots - y(x+a-1) = 0$$

ノ解ナルコトヲ知ル. 故ニソハ又

$$y(x) + y(x+2) + y(x+4) + \cdots + y(x+a-2) = 0$$

ノ解ナルコトヲ知ル. 而シテコハ又

$$p_0 + p_1 + p_2 + \cdots + p_n = 0,$$

$$p_0 - p_1 + p_2 - \cdots + (-1)^n p_n = 0$$

(1) 本論文初頁ノ脚註ニヨリ右邊ヲ 0 ノ代リニ λ ト置クコトヲ得.

即チ

$$\prod_{0, a-1}^k (p_0 + p_1 \sigma^k + p_2 \sigma^{2k} + \cdots + p_n \sigma^{nk})$$

ニ等シ. 但シ σ ハ $x^a - 1 = 0$ ノ根ナリ. 然ルニ前節ノ定理ノ假設ニヨリ此ノ乗積ノ各因數ハ 0 ニ等シカラス. 故ニ彼ノ輪換行列式ハ 0 ニ等シカラス. 故ニ上記ノ聯立方程式ガ成立スルニハ

$$y(x) = 0, \quad y(x+1) = 0, \quad y(x+2) = 0, \quad \cdots, \quad y(x+a-1) = 0.$$

故ニ定理ハ證明セラレタリ.

3. 格段ナル場合トシテ $n=2, a=3$ トシ又係數 p, q, r ヲ實數トシ函數方程式

$$py(x) + qy(x+1) + ry(x+2) = \lambda$$

ガ週期 3 ヲ有スル解ニヨリテ満足セラルル條件ヲ見出サントス.

$$(1) \quad p + q + r \neq 0.$$

此場合ハ $\frac{\lambda}{p+q+r} = \lambda'$ ト置キ第一節ノ如クセバ

$$p\{y(x) - \lambda'\} + q\{y(x+1) - \lambda'\} + r\{y(x+2) - \lambda'\} = 0.$$

故ニ

$$y(x) - \lambda' = y_1(x)$$

ト置ケバ

$$py_1(x) + qy_1(x+1) + ry_1(x+2) = 0.$$

此方程式ガ 0 ニアラザル連續的解ヲ有スルニハ兩代數方程式

$$1 - x^3 = 0,$$

$$p + qx + rx^2 = 0$$

ガ少クトモ一ツノ共通根ヲ有セザルベカラズ. 即チ ω ヲ 1 ノ立方基根 cubic primitive root トスレバ二方程式

$$p + q\omega + r\omega^2 = 0,$$

$$p + q\omega^2 + r\omega = 0$$

ノ中少クトモ一ツガ成立セザルベカラズ. 然ルニ p, q, r ハ實數ナリ. 故ニ

$$p = q = r.$$

實際此場合ニハ所題ノ方程式

$$y(x) + y(x+1) + y(x+2) = \frac{\lambda}{p}$$

ハ

$$y(x) = \frac{\lambda}{3p} + \cos \frac{2\pi}{3}x \cdot \Psi(x)$$

ナル解ヲ有ス。但シ $\Psi(x)$ ハ 1 ヲ週期トスル函數ニシテ λ ハ 0 ナルモ可ナリ。

$$(2) \quad p+q+r=0.$$

此場合ニハ窪田教授ノ示サル、所ト同様ニシテ (本誌第 11 卷第 153 頁) $\lambda \neq 0$ ナレバ全然解ナク $y(x) = \text{const.}$ タルコトサヘ能ハズ。

$\lambda=0$ トスレバ所題ノ方程式ハ

$$p\{y(x+1)-y(x)\} = r\{y(x+2)-y(x+1)\}.$$

x ノ代ハリニ $x+1, x+2$ ト置ケバ

$$p\{y(x+2)-y(x+1)\} = r\{y(x)-y(x+2)\},$$

$$p\{y(x)-y(x+2)\} = r\{y(x+1)-y(x)\}.$$

故ニ $y(x+1)-y(x)=0$ ナルコト能ハザレバ此三等式ヲ相乗ジテ

$$p^3 = r^3,$$

故ニ

$$p = r.$$

故ニ所題ノ方程式ニヨリ

$$y(x+1) - y(x)$$

ハ 1 ヲ週期トスル函數ナリ。今之ヲ $\Psi(x)$ ト置キ, $y(x) = \Psi(x) \cdot \varphi(x)$ トスレバ

$$\varphi(x+1) - \varphi(x) = 1,$$

故ニ

$$\varphi(x) = x + (\text{f. w. p. } 1),$$

故ニ

$$y(x) = \Psi(x) \cdot x + \Psi_1(x),$$

但シ $\Psi(x)$ ト $\Psi_1(x)$ トハ共ニ 1 ヲ週期トスル函數ナリ。定數ナルコト固ヨリ可ナリ。然レドモ斯クシテ得タルモノハ $\Psi(x)=0$ ナルニアラザレバ 3 ヲ週期トセズ。又 $\Psi(x)=0$ ナルモ 3 ヲ基週期トセズ。

故ニ次ノ定理ヲ得

係数 p, q, r ガ實數ナル函數方程式

$$py(x) + qy(x+1) + ry(x+2) = \lambda$$

ハ λ ノ如何ニ拘ハラズ

$$p = q = r$$

ナルニアラザレバ 3 ヲ基週期トスル連續的解ヲ有セズ. 而シテ其ノ時ニ於ケル格段ナル解ハ

$$\frac{\lambda}{3p} + \cos \frac{2\pi}{3} x. \text{ (f. w. p. 1)}$$

ナリ.

4. 次ギニ $n=2, a=4$ トシ又係数 p, q, r ヲ實數トシ函數方程式

$$py(x) + qy(x+1) + ry(x+2) = \lambda$$

ガ週期 4 ヲ有スル解ニヨリテ満足セラル、條件ヲ見出サントス.

(1) $p+q+r \neq 0$ トシ前節ノ如ク

$$\lambda \div (p+q+r) = \lambda',$$

$$y(x) - \lambda' = y_1(x)$$

ト置ケバ

$$py_1(x) + qy_1(x+1) + ry_1(x+2) = 0.$$

此ノ方程式ガ 0 ニアラザル連續解ヲ有スルニハ兩代數方程式

$$1 - x^4 = 0,$$

$$p + qx + rx^2 = 0$$

ガ少クトモ一ツノ共通根ヲ有セザルベカラズ. 故ニ

$$p - q + r = 0,$$

$$p + qi - r = 0,$$

$$p - qi + r = 0$$

ノ中少クトモ一ツガ成立セザルベカラズ. 此三ツ又ハ初メノ二ツ又ハ後ノ二ツガ同時ニ成立スルコトナシ. 殘ルハ次ノ場合ナリ

$$(a) \quad p - q + r = 0, \quad q \neq 0.$$

$$(b) \quad q = 0, \quad p = r.$$

$$(c) \quad q = 0, \quad p = -r.$$

第一ノ場合 (a) ニ於テハ q ノ代ハリニ $p+r$ ト置キテ

$$p\{y_1(x) + y_1(x+1)\} = -r\{y_1(x+1) + y_1(x+2)\}$$

$$p\{y_1(x+1)+y_1(x+2)\}=-r\{y_1(x+2)+y_1(x+3)\},$$

$$p\{y_1(x+2)+y_1(x+3)\}=-r\{y_1(x+3)+y_1(x)\},$$

$$p\{y_1(x+3)+y_1(x)\}=-r\{y_1(x)+y_1(x+1)\}.$$

故ニ $y_1(x)+y_1(x+1)=0$ ナルニアラザレバ相乗スルコトニヨリテ

$$p=\pm r.$$

若シ $p=r$ トスレバ $q=2p$. 故ニ所題ノ方程式ハ

$$y_1(x+1)+y_1(x+2)=-\{y_1(x)+y_1(x+1)\}$$

トナル. 故ニ

$$y_1(x)+y_1(x+1)=e^{\pi ix}. \text{ (f. w. p. 1).}$$

故ニ

$$y_1(x)=\varphi(x) e^{\pi ix}. \text{ (f. w. p. 1)}$$

ト置カバ

$$\varphi(x)-\varphi(x+1)=1,$$

故ニ

$$\varphi(x)=-x+\text{(f. w. p. 1)},$$

故ニ

$$y_1(x)=e^{\pi ix}\{-x\Psi(x)+\Psi_1(x)\},$$

但シ Ψ ト Ψ_1 トハ共ニ 1 ヲ週期トスル函數ナリ. 但シコハ 4 ヲ週期トセズ.

次ギニ $p=-r$ トスレバ $q=0$. 之レ假設ニ反ス.

$$\text{故ニ} \quad y_1(x)+y_1(x+1)=0$$

ナルコトヲ要ス. 故ニ

$$y_1(x)=e^{\pi ix}. \text{ (f. w. p. 1).}$$

コハ所望ニ適セス.

第二ノ場合 (b) ニ於テハ

$$p-q+r\neq 0.$$

故ニ此ノ條件ト $p+q+r\neq 0$ トヲ併セ考フレバ, 所設ノ方程式ニ解アルトキハ第 2 節ノ末端ノ注意ニヨリテ其ノ解ハ

$$y_1(x)+y_1(x+2)=0$$

ノ解ナリ. 故ニ

$$y_1(x)=e^{\frac{\pi}{2}ix}. \text{ (f. w. p. 2).}$$

コハ 4 ヲ基週期トシ所望ニ適ス.

第三ノ場合 (c) ニ於テハ $q=0$, $p=-r$. 故ニ所題ノ方程式ハ

$$y_1(x) - y_1(x+2) = 0.$$

故ニ

$$y_1(x) = f. w. p. 2.$$

コハ 4 ヲ基週期トセズ.

故ニ結局 $p+q+r \neq 0$ ナルトキニハ $q=0$, $p=r$ ノトキノミ 4 ヲ基週期トスル解ヲ有ス. ソハ

$$e^{\frac{\pi}{2}ix}. (f. w. p. 2)$$

ナリ.

$$(2) \quad p+q+r=0 \quad \text{トスレバ} \quad (1)$$

$$p\{y(x) - y(x+1)\} - r\{y(x+1) - y(x+2)\} = \lambda.$$

故ニ $p \neq r$ ナルトキニハ

$$\lambda \div (p-r) = \lambda'$$

及ビ

$$y(x) - y(x+1) - \lambda' = y_1(x)$$

ト置ケバ

$$py_1(x) = ry_1(x+1),$$

故ニ

$$y_1(x) = -\left(\frac{p}{r}\right)^x. (f. w. p. 1),$$

故ニ

$$y(x+1) - y(x) = \left(\frac{p}{r}\right)^x. (f. w. p. 1) - \lambda'.$$

是ニ於テ

$$y(x) = \varphi(x). \left(\frac{p}{r}\right)^x. (f. w. p. 1) - \lambda'x$$

ト置ケバ

$$p\varphi(x+1) - r\varphi(x) = r.$$

故ニ $r \neq 0$ ナルベケレバ

$$\varphi(x) = \left(\frac{r}{p}\right)^x (f. w. p. 1) + \frac{r}{p-r}.$$

(1) コハ亦第 3 節 (2) ノ如ク論ズルヲ得.

故ニ

$$y(x) = \mathcal{F}(x) + \frac{r}{p-r} \left(\frac{p}{r} \right)^x \mathcal{F}_1(x) - \lambda' x,$$

但シコハ 4 ヲ週期トナサズ。

次ギニ $p=r$ トスレバ $p \neq 0$ ナルベケレバ

$$\{y(x) - y(x+1)\} - \{y(x+1) - y(x+2)\} = \frac{\lambda}{p}.$$

故ニ

$$y(x+1) - y(x) = (\text{f. w. p. } 1) + \frac{\lambda}{p} x.$$

故ニ

$$y(x) = \varphi(x) \cdot (\text{f. w. p. } 1) + \frac{\lambda}{2p} x(x-1)$$

ト置ケバ

$$\varphi(x+1) - \varphi(x) = 1,$$

故ニ

$$\varphi(x) = x + (\text{f. w. p. } 1),$$

故ニ

$$y(x) = x \mathcal{F}(x) + \mathcal{F}_1(x) + \frac{\lambda}{2p} x(x-1).$$

之レ亦 4 ヲ週期トセズ。

故ニ次ノ定理ヲ得。

p, q, r ガ實數ナルトキ函數方程式

$$py(x) + qy(x+1) + ry(x+2) = \lambda$$

ガ 4 ヲ基週期トスル解ヲ有スルハ $q=0, p=r$ ノトキニ限ル。而シテ其ノ時ニ於ケル一般ナル解ハ

$$\frac{\lambda}{2p} + e^{\frac{\pi}{2}ix} \cdot (\text{f. w. p. } 2)$$

ナリ。

5. 第 1 節ニ於テハ有理週期 a ヲ基週期トシ、其ノ最低項ニ於テ表ハセル形ヲ a/β トシ、 a ガ n ヨリ大ナラザルトキハ ka ヲ n ヨリ大ナル如ク正ノ整數 k ヲ選ビテ進ムベシト論ジタリ。今斯クノ如キ k ヲ選バズシテ判定條件ヲ作ルコトモ容易ナリ。先ヅ

$$n = qa + r \quad (0 \leq r < a)$$

トシ q ト r ハ 0 又ハ正ノ整數トス. 然ラバ所題ノ函數方程式ハ

$$\begin{aligned} & (p_0 + p_\alpha + p_{2\alpha} + \cdots + p_{q\alpha})y(x) \\ & + (p_1 + p_{\alpha+1} + p_{2\alpha+1} + \cdots + p_{q\alpha+1})y(x+1) + \cdots \\ & + (p_r + p_{\alpha+r} + p_{2\alpha+r} + \cdots + p_{q\alpha+r})y(x+r) \\ & + (p_{r+1} + p_{\alpha+r+1} + p_{2\alpha+r+1} + \cdots + p_{(q-1)\alpha+r+1})y(x+r+1) \\ & + (p_{r+2} + p_{\alpha+r+2} + p_{2\alpha+r+2} + \cdots + p_{(q-1)\alpha+r+2})y(x+r+2) \\ & + \cdots \\ & + (p_{\alpha-1} + p_{2\alpha-1} + p_{3\alpha-1} + \cdots + p_{q\alpha-1})y(x+\alpha-1) = 0 \end{aligned}$$

トナスコトヲ得. 故ニ次ノ定理ヲ得. コレ第 1 節ノモノヲ包括スルモノナリ.

函數方程式

$$p_0 y(x) + p_1 y(x+1) + p_2 y(x+2) + \cdots + p_n y(x+n) = \lambda$$

ガ定數ノ外ニ既約有理數 α/β ヲ基週期トスル連續解ヲ有セザルニハ q ト r トヲ 0 又ハ正ノ整數トナシ $n = qa + r$ ($0 \leq r < q$) トナシタルトキ代數方程式

$$\begin{aligned} & (p_0 + p_\alpha + p_{2\alpha} + \cdots + p_{q\alpha}) + (p_1 + p_{\alpha+1} + p_{2\alpha+1} + \cdots + p_{q\alpha+1})x \\ & + \cdots + (p_r + p_{\alpha+r} + p_{2\alpha+r} + \cdots + p_{q\alpha+r})x^r \\ & + (p_{r+1} + p_{\alpha+r+1} + p_{2\alpha+r+1} + \cdots + p_{(q-1)\alpha+r+1})x^{r+1} \\ & + (p_{r+2} + p_{\alpha+r+2} + p_{2\alpha+r+2} + \cdots + p_{(q-1)\alpha+r+2})x^{r+2} \\ & + \cdots \\ & + (p_{\alpha-1} + p_{2\alpha-1} + p_{3\alpha-1} + \cdots + p_{q\alpha-1})x^{\alpha-1} = 0 \end{aligned}$$

ガ二項方程式

$$1 - x^\alpha = 0$$

ト共通根ヲ有セザレバ十分ナリ.

此定理ノ十分條件ハ格段ナル場合ニ於テ極メテ適用シ易シ.

例ヘバ p, q, r ガ實數ニシテ $p+q+r \neq 0$ ナルトキ函數方程式

$$py(x) + qy(x+1) + ry(x+2) = \lambda, \quad (p \neq 0, r \neq 0),$$

ガ定數ノ外 3 ヲ基週期トスル連續解ヲ有セザルニハ

$$\begin{aligned} p + qx + rx^2 &= 0, \\ 1 + x + x^2 &= 0 \end{aligned}$$

ガ共通根ヲ有セザレバ十分ナリ. 故ニ $p=q=r$ ナラザレバ十分ナリ (第 3 節參照).

又例ヘバ同假設ノ下ニ於テ同函數方程式ガ定數ノ外 4 ヲ基週期トスル連續解ヲ有セザルニハ

$$\begin{aligned} p+qx+rx^2 &= 0, \\ 1+x+x^2+x^3 &= 0 \end{aligned}$$

ガ共通根ヲ有セザレバ十分ナリ. 故ニ $p-q+r=0$ ナラザルカ又ハ $q=0, p=r$ ナラザルカナレバ十分ナリ (第 4 節參照).

p, q, r, s ガ實數ニシテ $p+q+r+s \neq 0$ ナルトキ函數方程式

$$py(x)+qy(x+1)+ry(x+2)+sy(x+3)=\lambda, \quad (p \neq 0, s \neq 0),$$

ガ定數ノ外 4 ヲ基週期トスル連續解ヲ有セザルニハ

$$\begin{aligned} p+qx+rx^2+sx^3 &= 0, \\ 1+x+x^2+x^3 &= 0 \end{aligned}$$

ガ共通根ヲ有セザレバ十分ナリ. 故ニ $p-q+r-s=0$ ナラザルカ又ハ $p=r, q=s$ ナラザルカナレバ十分ナリ.

前陳ノ一般定理ニ於テ a ガ素數ナルトキハ

$$1+x+x^2+\cdots+x^{a-1}=0$$

ハ有理數ノ範圍ニ於テ既約方程式⁽¹⁾ナルヲ以テ次ノ定理ヲ得.

$p_0, p_1, p_2, \dots, p_n$ ガ悉ク有理數ニシテ $p_0+p_1+p_2+\cdots+p_n \neq 0$ ナルトキ a ガ素數ナラバ, 所望ヲ果タスニハ

$$\begin{aligned} p_0+p_a &+ p_{2a} + \cdots + p_{qa}, \\ p_1+p_{a+1} &+ p_{2a+1} + \cdots + p_{qa+1}, \\ &\cdots \cdots \cdots \\ p_r+p_{a+r} &+ p_{2a+r} + \cdots + p_{qa+r}, \\ p_{r+1}+p_{a+r+1} &+ p_{2a+r+1} + \cdots + p_{(q-1)a+r+1}, \\ p_{r+2}+p_{a+r+2} &+ p_{2a+r+2} + \cdots + p_{(q-1)a+r+2}, \\ &\cdots \cdots \cdots \\ p_{a-1}+p_{2a-1} &+ p_{3a-1} + \cdots + p_{qa-1} \end{aligned}$$

(1) 有理數ノ範圍トイフ代リニ此ノ方程式ガ既約ナルガ如キ數ノ範圍トイヒ得ルコト勿論ナリ. 而シテ其ノトキニハ $p_0, p_1, p_2, \dots, p_n$ ガ其ノ數ノ範圍ニ屬スレバ上ノ定理ヲ得ベシ.

ガ悉ク相等シキニアラザレバ十分ナリ.

例ヘバ p_0, p_1, p_2, p_3, p_4 ガ有理數ナルトキ $p_0 + p_3 = p_1 + p_4 = p_2$ ナラザレバ函數方程式

$$p_0 y(x) + p_1 y(x+1) + p_2 y(x+2) + p_3 y(x+3) + p_4 y(x+4) = \lambda$$

ガ定數ノ外ニ 3 ヲ基週期トスル連續解ヲ有セズ.

又次ノ定理ヲ得.

$p_0, p_1, p_2, \dots, p_n$ ガ悉ク有理數ニシテ $p_0 + p_1 + p_2 + \dots + p_n \neq 0$ ナルトキ, $n+1$ ガ素數ナラバ函數方程式

$$p_0 y(x) + p_1 y(x+1) + p_2 y(x+2) + \dots + p_n y(x+n) = \lambda$$

ガ定數ノ外ニ $n+1$ ヲ基週期トスル連續解ヲ有セザルニハ

$$p_0 = p_1 = p_2 = \dots = p_n$$

ナラザレバ十分ナリ. 而シテ $p_0 = p_1 = p_2 = \dots = p_n$ ナルトキニハ

$$\frac{\lambda}{(n+1)p} + \cos \frac{2\pi}{n+1} x. \text{ (f. w. p. 1)}$$

ナル解ヲ有ス.

更ニ次ノ定理ヲ得.

$p_0, p_1, p_2, \dots, p_r$ ガ悉ク有理數ニシテ $p_0 + p_1 + p_2 + \dots + p_r \neq 0$ ナルトキ, $n+1$ ガ $r+1$ ヨリ大ナル素數ナラバ函數方程式

$$p_0 y(x) + p_1 y(x+1) + p_2 y(x+2) + \dots + p_r y(x+r) = \lambda$$

ハ定數ノ外ニ $n+1$ ヲ基週期トスル連續解ヲ有セズ.

大 正 六 年 七 月

抄 錄 短 評

I. 新 刊 書 目

E. BOUTROUX, *Natural law in science and philosophy*. Authorized translation by F. Rothwell. New York, Macmillan, 1914. 218 p.

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G. DI DIA. *L'omografia e l'involuzione nelle forme fondamentali di la specie e la dualità nella geometria proiettiva*. Bologna, tip. Cuppini, 1917. L. 1.05.

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K. WHITED. *Trisection of an angle*. The problem solved geometrically with other problems. Redmond, Ore., K. Whited, 1917. 14 p. \$ 1.00.

S. P. THOMPSON. ("F. R. S."). *Calculus made easy; being a very simplest introduction to those beautiful methods of reckoning which are generally called by the terrifying names of the differential calculus and the integral calculus*. Second edition, enlarged. London, Macmillan, 1917. 2+265 p. 2s.

E. ZONADARI. *Integrazione grafica e studio delle equazioni differenziali ordinarie del primo ordine coi metodi della geometria descrittiva*. Milano, Soc. ed. Dante Alighieri (Roma, tip. Nazionale, Bertero), 1917. 9+112 p. L. 3.50.

W. N. Rose. Mathematics for engineers, Part 1. London, Chapman and Hall, Ltd., 1918. 14+510 p. net 8s. 6d.

大戦争ノ影響ヲ受ケテ工業教育ノ勃興ヲ極ムル今日、網羅的ニシテシカモ摘要簡明的ナル数学書ヲ有スルコトハ急務中ノ急務タリ。本書ハ Chapman and Hall ガ W. J. Lineham 氏ヲ監修者トシテ發行スル The directly-useful technical series ノ一篇トシテ出版セラレタルモノニシテ Elementary and higher algebra, mensuration and graphs, and plane trigonometry ヲ包含セリ。Part 2. ハ Differential and integral Calculus, practical applications of the calculus, polar coordinates, differential equations, harmonic analysis, spherical trigonometry, vector analysis, mathematical probability etc. ヲ包含スル筈ナリ。既刊ノ Part 1 ニ就キテ見ルニ上述ノ要求ヲ充填スルニ最も適切ナルヲ認ム。工業数学ヲ學ベントスル學生又ハ之ヲ毎日ノ作業ニ於テ緊要トスル工業家ニ推奨スベキ良書ナリ。(T. H.)

II. 雜誌内容

下記ノ雑誌ニ掲載セラレタル論文中、数学又ハ数理物理学ニ關係ナキモノハ、省略ス。

Proceedings of the Edinburgh Mathematical Society, Vol. 35, Part 1, 1917.

E. T. Whittaker, On the latent roots of compound determinants and Brill's determinants. Wm. P. Milne, The apolar locus of two tetrads of points. The co-apolar of a cubic curve. G. B. Jeffery, Transformations of axes for Whittaker's solution of Laplace's equation. L. R. Ford, On a class of continued fractions. H. Datta, On the theory of continued fractions.

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W. Hope-Jones, The principles of probability and approximations in arithmetic.

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S. Ramanujan, On the expression of a number in the form $ax^2+by^2+cz^2+dw^2$. C. E. Van Horn, An axiom in symbolic logic. J. G. P. Nicod, A reduction in the number of the primitive propositions of logic. G. N. Watson, Bessel functions of equal order and argument. G. N. Watson, The limits of applicability of the principle of stationary phase. H. C. Pocklington, The direct solution of the quadratic and cubic binomial congruences. G. H. Hardy, On a theorem of Mr. G. Pólya. G. H. Hardy, On the convergence of certain multiple series. G. N. Watson, Bessel functions of large order. H. J. Mordell, On Mr. Ramanujan's empirical expansions of modular functions.

The messenger of mathematics, Vol. 47, No. 4, Aug., 1917.

E. T. Bell, Fourier series for the squares of Hermite's eighteen doubly periodic theta quotients. F. H. Jackson, The q -Integral analogous to Borel's integral.

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R. D. Carmichael, Comparison theorems for homogeneous linear differential equations of general order. H. B. Fine, Note on a substitute for Duhamel's theorem. J. H. Weaver, Some properties of a straight line and circle and their associated parabolas. C. L. E. Moore, Motions in hyperspace. J. R. Kline, A definition of sense on closed curves in non-metrical plane analysis situs. O. E. Glenn, Covariant expansion of a

modular form. G. M. Green, The intersections of a straight line and hyperquadric. E. T. Bell, Numerical functions of $[x]$. L. P. Eisenhart, Surfaces which can be generated in more than one way by the motion of an invariable curve.

Bulletin of the American Mathematical Society, Vol. 24, No. 5-6, Feb.-March, 1918.

O. D. Kellogg, The eleventh regular meeting of the Southwestern Section. G. M. Green, Note on conjugate nets with equal point invariants. J. F. Ritt, On the differentiability of asymptotic series. L. P. Eisenhart, Darboux's contribution to geometry. F. N. Cole, The twenty fourth annual meeting of the American Mathematical Society. A. Dresden, The Winter-meeting of the Chicago Section. R. D. Carmichael, Elementary inequalities for the roots of an algebraic equation. H. Bateman, The solution of the wave equation by means of definite integrals.

Mathematische Annalen, Bd. 77, Heft 3-4.

H. Weyl, Ueber die Gleichverteilung von Zahlen mod. Eins. M. Bauer, Zur Theorie der algebr. Zahlkörper. M. Bauer, Ueber zusammengesetzte Zahlkörper. B. von Ludwig, Ueber eindeutige Umkehrbarkeit Abelscher Integrale. O. Blumenthal, Einige Minimums-Sätze über trigonometrische und rationale Polynome. O. Muehlendyck, Ueber die regulären eindimensionalen analytischen Somenmannigfaltigkeiten. J. v. S. Nagy, Ueber die reellen Züge ebener und Raum-Kurven. F. Hausdorff, Die Mächtigkeit der Borelschen Mengen. K. Knopp, Bemerkungen zur Struktur einer linearen perfekten nirgends dichten Punktmenge. D. König, Ueber Graphen u. ihre Anwendung auf Determinantentheorie u. Mengenlehre. M. Dehn, Ueber die Starrheit konvexer Polyeder. W. Künstermann, Funktionen von beschränkter Schwankung in zwei reellen Veränderlichen. O. Szasz, Ueber die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen. G. Pólya, Ueber Potenzreihen mit ganzzahligen Koeffizienten. E. Hilb, Zur Theorie der linearen Integrodifferentialgleichungen. Emmy Noether, Die Funktionalgleichung der isomorphen Abbildung. A. Speiser, Gruppendeterminanten und Körperdiskriminante. G. Voghera, Ein direkter Beweis für die Normalform der komplexen Zahlensysteme.

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B. Kerst, Ueber Polyeder, deren Netze durch konvexe Polygone gebildet werden. E. Haentzchel, Eine artilleristische Aufgabe. Pyrkosch, Die mathematischen Reformbestrebungen. Max Brües, Mittelpunktswinkel, Umfangswinkel, Sehne und Kreisviereck in allgemeinsten Behandlung. K. Riedir, Zur Einführung des Logarithmus im Kleinschen Sinne. O. Lesser, Zur Behandlung der Kegelschnitte. O. Herrmann, Einige Gruppen von geometrischen Aufgaben, die auf unbestimmte Ausdrücke führen. P. Kiesling, Ueber die Kurve der Schattenenden des Gnomons. K. Quensen, Konstruktion der komplexen Wurzeln von Gleichungen zweiten und dritten Grades. N. Gennimatas, Methodische Bemerkungen zur ebenen Trigonometrie. v. Sanden, Vektorenrechnung und analytische Geometrie.

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C. Burstin, Die Spaltung des Kontinuums in κ , überall dichte Mengen. E. Dolezal, Das Rückwärtseinschneiden auf der Sphäre, gelöst auf photogrammetrischen Wege. II. Das Pantograph-Planimeter. W. Gross, Zur Poisson'schen Summierung. G. Kowa-

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Nouvelles Annales de Mathématiques, (4) T. 17, Déc., 1917, T. 18, Jan., 1918.

J. B. Pomey, Sur une propriété de la fraction rationnelle du second degré. Ch. Michel, Développantes et développées aréolaires. G. Fontené, Identités à démontrer. E. Cahen, Remarques sur un article de M. Mathieu Weill. H. Lebesgue, Sur deux théorèmes de Mannheim et de M. Bricard concernant les lignes de courbures et les lignes géodésiques de quadriques. T. Hayashi, Le produit de cinq nombres entiers consécutifs n'est pas le carré d'un nombre entier. J. Bouchary, Analogies entre le triangle et le quadrilatère. J. Juhel-Rénoy, Sur les foyers de courbes planes. R. Bouvaist, Sur deux propositions de Ribaucour. C. H. Sisam, Sur l'ordre de surfaces engendrées par courbes d'un ordre donné.

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H. Vergne, Sur les équations générales de la mécanique analytique(fin). A. Buhl, Sur les sommes abéliennes de volumes cyclidoconiques. M. Brillouin, Sources électromagnétiques dans les milieux uniaxes.

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S. Lefschetz, Sur les intégrales doubles des variété algébriques. U. Dini, Sugli sviluppi in serie $\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos \lambda_n x + b_n \sin \lambda_n x)$ dove le λ_n sono radici della equazione trascendente $F(z) \cos \pi z + F_1(z) \sin \pi z = 0$. A. F. Carpenter, Some fundamental relations in the projective differential geometry of ruled surfaces.

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G. Usai, Una questione di analisi combinatoria (continuazione). S. W. Reaves, Metric properties of flecnodes on ruled surfaces. V. Gallico, Sulle condizioni iniziali che determinano gli integrali delle equazioni differenziali ordinarie.

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C. Rosati, Sulle valenze delle corrispondenze algebriche fra i punti di una curva algebrica. A. Pensa, Su alcune omografie speciali e sugli operatori omografici C, R. A. Pensa, Sull'operatore omografico R'.

Bulletin de l'Académie des Sciences de Russie, (6), No. 1-4, 1918.

N. Kryloff et J. Tamarkine, Sur la méthode de W. Ritz pour la solution approchée des problèmes de la physique mathématique. W. Stekloff, Remarques sur les quadratures.

雜 錄 彙 報

歐 米 諸 大 學 ノ 課 程

北 米 合 衆 國

こ ら む び あ 大 學 (1917-18)

ふいすく Fiske, 微分方程式 (4). こゝる Cole, 群論 (3). 不變式論及ビ高次平面曲線論 (前半年間, 4). まくれ J. MacLay, 幾何學作圖論 (前半年間, 3). 橢圓函數論 (前半年間, 3). かいざ Keyser, 幾何學 = 於ケル近世ノ理論 (4). 數學 (後半年間, 3). すみす D. E. Smith, 數學史 (2). かすな Kasner, 微分幾何ノせみなり (2). 函數方程式及ビ積分方程式 (前半年間, 3). ふあいと W. B. Fite, 微分方程式 (後半年間, 3). ほくす H. E. Hawkes, 曲線ノ微分幾何學 (後半年間, 3).

か い ね る 大 學 (1917-18)

まくまほん J. McMahon, 確カラシサノ理論 (3). 保險學序論 (3). すないだ Snyder, 射影幾何學 (3). しゃいぶ Sharpe, 動徑解析及ビ其物理學上ノ應用 (第一學期間, 3). かーヴあ Carver, 初等群論 (第二學期間, 3). 高等數學綱要 (しるづあしまんト共同, 3). らなむ Ranum, 微分幾何學 (第一學期間, 3). ぎれすび D. C. Gillespie, 高等微積分學 (3). ふーるゐつつ W. A. Hurwitz, 物理學上ノ微分方程式 (3). くれーぐ Craig, ふーりえノ級數及ビポテンシアル函數 (3). 數學師範課程 (3). おーうえんす F. W. Owens, 數理物理學 (3). しるづあしまん Silverman, 無限級數 (3). まつけるぐい McKelvey, 代數曲線 (3). べつつ Betz, 初等微分方程式 (3). がば M. G. Gaba, 方程式論 (第一學期間, 3). ぎるまん Gilman, 高等解析幾何學 (3).

は い づ あ い ど 大 學 (1917-18)

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猶ホ二週間=亘ルばいこふノ解析學=關スルせみなりアリ。又研究課トシテおすぐつどノ函數論, ぼいしゃノ線狀微分方程式ノ實解, ぶーとんノ點變換論, くりりっちノ幾何學, ばいこふノ微分方程式, ちゃくそんノ實變數ノ函數論, ぐりーんノ微分幾何學アルベシ。

い り の い づ 大 學 (1917-18)

たうんせんど E. J. Townsend, 複素變數ノ函數論 (3). 微分方程式及ビ高等微積分學 (3). みらい G. A. Miller, 初等群論 (3). 方程式論及ビ行列式 (第一學期間, 3). リーッ

Rietz, 統計學 (3). しょう J. B. Shaw, 一般代數學 (3). さいざむ Sisam, 代數表面, 立體解析幾何學 (第二學期間, 3). えむひ Emch, 射影幾何學, 作圖幾何學 (第二學期間, 3). かみけい Carnichael, 線状有限差方程式ノ理論 (3). くらとほ Crathorne, 數學機械 (第二學期間, 3). りとる Lytle, 師範課程 (第一學期間, 2). 數學史 (第二學期間, 2). けむぶな Kempner, 近世代數學 (3).

じょんほぶきんす大學 (1917-18)

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べんしるづゐにや大學 (1917-18)

くろれ Crawley, 高次平面曲線 (2). ふいしゃ Fisher, 複素變數ノ函數論 (2). しわっと I. J. Schwatt, 無限級數, 無限乘積 (2). ほれっと Hallett, 有限群 (2). さふさど Safford, 偏微分方程式 (2). ばぶ Babb, 數論 (2). ちえむばす Chambers, 綜合射影幾何學 (2). れん Glenn, 變分學 (第二學期間, 2). みちえる Mitchell, 代數的數ノ理論 (2). むあ R. L. Moore, 數學ノ基礎 (2). べい F. W. Beal, 微分幾何 (2).

ぷりんすとん大學 (1917-18)

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えいる大學 (1917-18)

ぶらうん E. W. Brown, 高等微積分學 (3). 高等力學 (2). ぴあぼんと Pierpont, 複素變數ノ函數論 (2). 楕圓函數 (2). すみす P. F. Smith, 微分方程式 (2). ろんぐれ Longley, 積分方程式 (第二學期間, 2). ポテンシャル論及ビ調和解析 (第一學期間, 2). まいる E. J. Miles, 變分學 (2). とれいし J. I. Tracy, 近世解析幾何學 (2). ばろ D. F. Barrow, 高等代數學 (2). くらむ W. L. Crum, 靜力學及ビ動力學 (2). ほういつともあ J. K. Whittmore, 微分幾何學 (2).

佛 蘭 西

College de France. (1917-18).

おむべいあ Humbert, 二次形式ニ關スル諸問題. あだま Hadamard, 線状二次偏微分方程式. ぶりのあん Brillouin, 緯度變化, 地球ノ構造及ビ一般運動ニ關スル諸結果, 之ニ關スル力學ノ問題. らんぢゅゐん Langevin, 相對律及ビ重力論.

巴里學士院ニ於ケル彈道學研究委員會ノ成績

巴里學士院ニ於テハ時局ニ鑑ミル所アリテカ Appell 教授ヲ委員長トシタル Commission de Balistique ナルモノガ設ケラレタ. 其研究成績ノ題目ハ時々 Comptes Rendus 誌上ニテ發表セラル、ガ其内容ハ固ク秘セラレテオル昨年中ノ研究題目ヲ舉ゲルト下ノ如クデアル. イサ、カ參考ニ資スル所ガアラウ.

Parodi, Sur une nouvelle méthode d'intégration mécanique de l'équation balistique.

Kampé de Fériet, Calcul des coefficients différentiels en un point d'une trajectoire.

Haag, Calcul des coefficients différentiels en un point d'une trajectoire.

M. Saint-Léon, Sur un projet de projectile.

Drach, Sur l'équation différentielle de la balistique extérieure et son intégration par quadratures.

Denjoy, Sur l'équation de la balistique extérieure.

Esclangon, Sur l'enregistrement pulsométrique des coups de canon.

Parodi, Sur le calcul numérique des trajectoires par arcs successifs.

Risser, Note relative à la recherche de la variation de portée résultant de l'effet du vent.

Esclangon, Sur certaines phénomènes de condensation qui accompagnent les projectile en marche; le problème de la stabibisation de certaines projectiles.

Esclangon, Sur l'expansion des gaz à la bouche des canon.

Kampé de Fériet, Sur l'expression par une fonction hypergéométrique de l'intégrale $\xi_n(\tau)$ qui s'introduit dans l'équation de l'hodographe, quand on suppose la résistance de l'aire de la form kv^n .

Garnier, Sur les valeurs limites de certaines coefficients différentielles d'une trajectoire balistique.

π ノ 近 似 値

L'Intermédiaire des Mathématiciens ノ 本年一二月號 p. 2 = F. Balitrand ナル人ハ

$$\pi = \left(\frac{4}{3}\right)^4$$

ナルコトヲ示セリ. 3, 4 ト云フ簡單ナル數ニテ π ヲ表ハシ得ルコト妙ナラズヤ.

二三雜誌中ノ注目スベキ論說記事

東京物理學校雜誌, 大正 7 年 3 月號及 4 月號

兩脚器ノミニテ作圖題ヲ解クコト

數理雜俎

或ル微分方程式ノ積分曲線ニ就テ

哲學雜誌, 大正 7 年 3 月號及 4 月號

否定ノ研究

幾何學ノ論理の基礎 (完)

理學界, 大正 7 年 3 月號

數學ノ新定理

保險雜誌, 大正 7 年 2 月號

我國ニ於ケル生命保險ノ保險料ニ就テ

年金ノ利率ヲ求ムル公式

歴史ト地理, 大正 7 年 2 月號

石黒信由

史學雜誌, 大正 7 年 2 月號及 3 月號

伊能忠敬ガ測地事業ニ成功シタル所以 (詳説)

日本數學發達ノ由來 (詳説)

理學士 黒河龍三氏

柳原吉次氏

理學士 黒須康之介氏

文學士 今福忍氏

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牧野信之助氏

理學士 大谷亮吉氏

三上義夫氏

東京數學物理學會年會

東京數學物理學會ニテハ四月一日ヨリ三日ニ至ル間年會ヲ開ケリ。今其ノ講演ノ題目ヲ列擧スレバ次ノ如シ。

- [第一日] 1. 志田順君、松山基範君、佐藤覺君：i. 地震初動ニ關スル研究（第二回報告）。ii. 地震ノ様式、附「マイクロバログラフ」ノ一案。2. 早乙女清房君：i. 太陽ノ分光寫眞ニ就テ。ii. 大正六年八月中ノ太陽ノ活勢。3. 中村左衛門太郎君：東京ニ於ケル地磁氣觀測ノ結果。4. 寺田寅彦君：i. Solar period of the magnetic pulsation. ii. Meteorological notes. 5. 築地克雄君：再ビ水平暈ニ就テ。6. 新城新藏君：On the masses and angular momenta of star system. 7. 本多光太郎君、大久保準三君：On a molecular theory of magnetism explaining the ferro-, para- and diamagnetic properties of matter. 8. 高嶺俊夫君、國分襄君：i. The Stark effect on calcium and magnesium lines. ii. Further studies on the Stark effect in helium and hydrogen. 9. 石原純君：i. Stark 効果ニ就テ。ii. 萬有引力現象ト光現象トノ關係。iii. 萬有引力ニ於ケル時間及空間ノ概念。10. 本多光太郎君：A note on the Weiss molecular field. 11. 長岡半太郎君：On the accuracy of the formulae for series spectra. 12. 日下部四郎太君：Quantum theory of time. [第二日] 13. 掛谷宗一君：On differential inequalities. 14. 龜田豐治朗君：Neue Begründung der Theorie der Fourierschen Reihe. 15. 小倉金之助君：i. On the striped net of curves without ambages in Dynamics. ii. On a generalisation of the Bonnet-Darboux theorem concerning the line of striction. iii. Theory of the point line connex (1, 1) in space. 16. 國枝元治君：Asymptotic formulae for coefficients of certain power series. 17. 寺澤寛一君：彈性體ノ平衡ニ關スル或問題。18. 愛知敬一君：Strength of material ノ問題二三。19. 寺田寅彦君：i. Instability of liquid film. ii. Structure of spark. 20. 本多光太郎君：i. A theory of Invar. ii. Latent heat of melting as the energy of molecular rotation. 21. 本多光太郎君、高木弘君：On a new magnetic steel. 22. 中村清二君：Thermal expansion coefficient of quartz. 23. 宗正路君：Thermal expansion and contraction of glass. 24. 菊池泰二君：真空内ノ壓力ニ就テ。25. 眞島正市君：爆發性瓦斯ノ二三ノ性質。26. 小野澄之助君：X 線ノ寫眞作用ト溫度トノ關係ニ就テ。27. 池内本君：Stereoscopic photographs of the tracks of α particles in air. 28. 小幡重一君：i. 電氣試驗所電壓標準ト Bureau of Standards 及 National Physical Laboratory 電壓標準トノ比較。ii. Note on the accuracy of copper voltameter. iii. Determination of E. M. F. of Weston normal cells by the Richards form silver voltameter. 29. 乙部孝吉君：簡單ナル kite line ノ equation. [第三日] 30. 本多光太郎君：On the thermal and electric conductivities of iron-nickel and iron-cobalt alloys. 31. 大河内正敏君、佐藤兌君：金屬ノ燒キ入レ燒キ鈍シニ伴フ熱膨脹率。32. 大河内正敏君、眞島正市君：i. 熔融金屬ヴィスコシティニ就テ。ii. モリブデン鋼ノ物理的性質ニ就テ。33. 松下徳次郎君：On slow contraction of quenched steel. 34. 本多光太郎君、村上武次郎君：On the structure of tungsten steel and that of chromium steel. 35. 堤秀夫君：On the change of electric resistance of metals during melting. 26. 田中正平君：繼續著音器。

各種賞典ノ授受

巴里科學院ハ、流體力學上ノ業績ニ對シテ Francœur 賞金 1000 ふらんヲ Henri Vaillat ニ授ケ、Étude des formes binaires non quadratiques à indéterminées réelles, ou complexes, ou à indéterminées conjuguées ニ對シテ Bordin 賞金 3000 ふらんヲ歩兵第三十四聯隊付少尉 Gaston Julia ニ授ケタリ。コハ非二次形式ノ算術的理論ニ於ケル何事カノ完成ニ關シテ懸賞問題トシテ提出セラレタルモノナリ。又剛體運動ノ幾何學的理論ニ關スル業績ニ對

シテ Montyon 賞金 700 ふらんヲ René de Saussure =授ケ、應用力學及ビ特ニ時刻測定ニ關
スル業績ニ對シテ Poncelet 賞金 2000 ふらんヲ Jules Andrade =授ケ、P. Duhem =ハ其
述作ノ全部、特ニ Le System du monde =對シテ Prix Petit d'Ormay des Sciences Mathé-
matiques 10,000 ふらんヲ授ケ、Vallée Poussin =ハ其著“極微解析學教程”及ビ“るべし
積分及ビ集合ノ函數”ニ對シテ、Parville ノ賞金 2000 ふらんヲ授ケ、Gomes Teixeira =
ハ(Obras sobre Mathematica =對シテ Binoux 賞金 2000 ふらんヲ授ケ、H. Lebesgue =ハ
微積分學ノ原理ニ關スル業績ニ對シテ Saintour 賞金 3000 ふらんヲ授ケタリ。

正 誤

本卷第 250 頁ニ載セタル素數ニ關スル定理ニ於テハ $n \equiv 1 \pmod{3}$ トナスベク、又報告者ハ
上田光雄氏ナリ。

諸 學 者 ノ 消 息

伊太利ノミラノ大學ノヴネツテ教授 Vito Volterra ハ巴里學士院ノ Associé étranger ニ選
舉セラレタリ。

佛蘭西バリエール學士院ノイテイニ紀念資金取扱委員三名ノ内數學部ニテノじおるだん教授 C.
Jordan ヲ推選セリ。

北米合衆國ニ於テハミシゴ大学ノクリッチ教授 J. L. Coolidge、いりのいず大学ノい
ぢんとん W. E. Edington、はしばしど大学ノふをいど Dr. L. R. Ford、いりのいず大学ノけ
るす J. M. Kells、はしばしど大学ノうえるしゆ J. L. Walsh 等ハ national military service
ニスレリ。

第一高等學校講師黑河龍三氏ハ同校教授トナレリ。

北米合衆國しらすきゆす大学ノでつか F. F. Decker ハ同大学教授トナリ、じおるだんず
Dr. J. L. Jones ハ助教授トナレリ。

北米合衆國いりのいず大学ニテうゐりあむぢゆいうえる College ノまつくあち Dr. J. E.
McAtee 及かんさす大学ノすていむり L. L. Steimley ハ數學科ノ教師トナリ、むっせるまん
Dr. J. R. Musselman ハわしんとん政府ノ一部ニ於テ統計學ヲ教授セシガ今回ソノ職ヲ止メ
タリ。

べるりん大学教授、べるりん學士院會員ふるべにゆす氏 G. Frobenius ハ 1917 年 8 月
3 日 67 歳ニテ死亡セリ。

べるりん大学教授、べるりん學士院會員へるめると氏 F. R. Helmert ハ 1917 年 6 月 15
日 73 歳ニテ死亡セリ。

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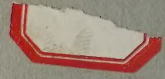
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